W. Dijkstra and R.M.M. Mattheij / Electronic Journal of Boundary Elements, Vol. 4, No. 2, pp. 67-81 (2006)

The condition number of the BEM-matrix arising from Laplace's equation

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Abstract

We investigate the condition number of the matrices that appear in the boundary element method. In particular we consider the Laplace equation with mixed boundary conditions. For Dirichlet boundary conditions, the condition number of the system matrix increases linearly with the number of boundary elements. We extend the research and search for a relation between the condition number and the number of elements in the case of mixed boundary conditions. In the case of a circular domain, we derive an estimate for the condition number of the system matrix. This matrix consists of two blocks, each block originating from a well-conditioned matrix. We show that the block matrix is also well-conditioned.

1 Introduction

Boundary Value Problems (BVP) can be solved by the Boundary Element Method (BEM). This method transforms the BVP into an integral equation, which after discretization of the boundary, results into a set of linear algebraic equations. The system matrix that appears in this set of algebraic equations is dense. It is a well-known fact that the condition number of the system matrix is at least order N, where N is the number of boundary elements, [1]. In [2] the Dirichlet BVP for the Laplace equation is studied, where two domains are taken into account: a circle and an ellipse. In both cases analytical expressions for the condition number are derived. The results show how the condition number depends on the radius of the circle or the aspect ratio of the ellipse. Comparable results are given in [3]. In [4] and [5] the Laplace equation on a circle with Dirichlet boundary conditions is again investigated. Special attention is given to the so called local condition number. All papers mentioned above deal with BVPs with Dirichlet boundary conditions. In boundary integral equations single and double layer potentials arise. The analysis of these boundary integral operators is a well-chartered area. For instance, in [6] the spectral properties of the Laplace and Helmholtz integral operators are investigated as well as the eigenvalues of the corresponding discrete operators. Preconditioning the matrices in the

BEM is the topic of [7], [8], and [9]. These papers also deal with the spectral properties of the boundary integral operators. In [10] the authors investigate Neuman's method, which can handle both Dirichlet and Neuman BVP's.

For a Laplace equation with either Dirichlet or Neuman boundary conditions the spectral properties of the single and double layer potentials have been investigated thoroughly. Yet the conditioning of the resulting algebraic system received little attention. For a problem with partial Dirichlet and partial Neuman boundary conditions this is even more true. It is well known that the matrices corresponding to the single-layer potential and double-layer potential are well conditioned, see e.g. [6] and [11]. The matrix for the mixed problem is composed of two blocks from these matrices. These blocks may be skew. Hence the question arises whether the system matrix for the mixed problem is still a reasonably conditioned matrix. In practical applications the possibility of an ill-conditioned matrix is never considered. This is therefore the topic of this paper: can we foresee ill-conditioned matrices in the mixed problem.

We do not consider the problem in all its generality, but restrict ourselves to a two-dimensional situation and a fairly simple geometry, a circle, so that we can give explicit expressions, or at least sharp estimates, for the underlying eigensystems. We are aware of several formulations of the boundary integral equations. In this paper we choose for the direct symmetric collocation formulation. The direct formulation involves functions that can be easily related to physical quantities, whereas the indirect formulation uses auxiliary functions that have no physical meaning. The symmetric formulation, involving the single and double layer potentials, is more commonly used than the non-symmetric formulation, which incorporates the hypersingular operator. We prefer the collocation method above the Galerkin method. Again the collocation method is more commonly used and it does not require a second integration step like Galerkin method does.

The paper is set up as follows. In Section 2 we introduce the BVP and we briefly explain how a set of linear algebraic equations is obtained. Section 3 describes the derivation of the estimate for the condition number. The estimate is shown to be fairly accurate for several examples, in Section 4.

2 Setting

In this section we briefly describe how a set of algebraic equations is obtained from a boundary value problem. In particular we consider the Laplace equation for the unknown function $u = u(\boldsymbol{x})$. The taken domain is a circle Ω with radius R. The boundary Γ of the circle is divided into two parts, i.e. $\Gamma = \Gamma_u \bigcup \Gamma_q$, as shown in Figure 1. On the part Γ_u we pose Dirichlet boundary conditions; on the part Γ_q we pose Neuman boundary conditions. We introduce $q := \partial u/\partial n$ as the normal derivative of u on the boundary and obtain the boundary value problem

$$\left\{ \begin{array}{rcll} \Delta u & = & 0, & \boldsymbol{x} \in \Omega, \\ u & = & \tilde{u}, & \boldsymbol{x} \in \Gamma_u, \\ q & = & \tilde{q}, & \boldsymbol{x} \in \Gamma_q, \end{array} \right. \tag{1}$$

where $\tilde{u}$ and $\tilde{q}$ represent the given boundary values. We use the BEM to find the unknown values of u on Γ_q and the unknown values of q on Γ_u.

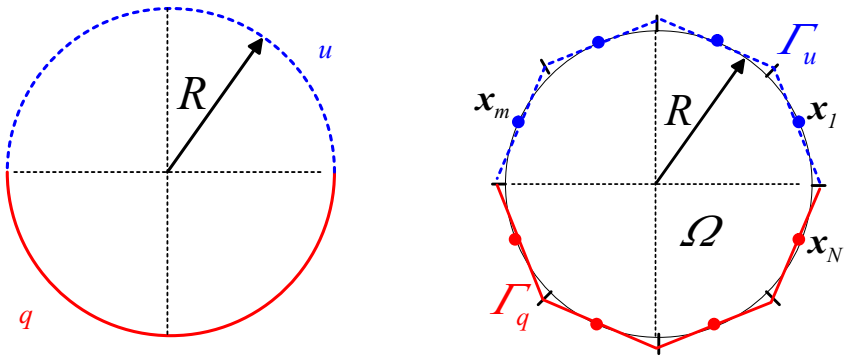

Figure 1: *The domain on which the Laplace equation holds. The boundary is divided into two parts and discretized into N elements: m elements have Dirichlet conditions and $N - m$ elements have Neuman conditions.*

Using the fundamental solution for the Laplace operator and Greens second identity, the boundary value problem transforms in the following boundary integral equation (cf [12]),

$$\Big(\frac{1}{2}\mathcal{I} + \mathcal{K}^d\Big)u = \mathcal{K}^s q, \tag{2}$$

where the integral operators are given by

$$\begin{aligned}\big(\mathcal{K}^s q\big)(\boldsymbol{x}_P) &:= -\frac{1}{2\pi}\int_\Gamma \ln\|\boldsymbol{x}_P - \boldsymbol{x}_Q\| q(\boldsymbol{x}_Q) d\Gamma_Q,\ \boldsymbol{x}_P \in \Gamma,\\ \big(\mathcal{K}^d u\big)(\boldsymbol{x}_P) &:= \frac{1}{2\pi}\int_\Gamma \frac{(\boldsymbol{x}_P - \boldsymbol{x}_Q, \boldsymbol{n}_Q)}{\|\boldsymbol{x}_P - \boldsymbol{x}_Q\|^2} u(\boldsymbol{x}_Q) d\Gamma_Q,\ \boldsymbol{x}_P \in \Gamma.\end{aligned} \tag{3}$$

Here $\boldsymbol{n}_Q$ is the outward normal on Γ at the boundary point $\boldsymbol{x}_Q$, $(\cdot,\cdot)$ is the standard Euclidean inner product and $\|\cdot\|$ the associated norm. The eigenvalues and eigenfunctions of the operators are given in Table 1 (cf [3], [6]). Note that the integral operators $\mathcal{K}^s$ and $\mathcal{K}^d$ and the Laplace operator Δ have the same eigenfunctions while the eigenvalues differ.

We approximate the boundary by N equispaced linear elements Γ_k, $k = 1, \ldots, N$, see Figure 1. We choose the first m elements to relate to the Dirichlet boundary and the last $N - m$ elements to the Neuman boundary. At the center of an element Γ_k we choose the nodal point $\boldsymbol{x}_k := [R\cos\theta_k, R\sin\theta_k]^T$, where the angle θ_k is given by $\theta_k := \pi/N + (k-1)2\pi/N$, $k = 1, \ldots, N$. At each element we approximate the functions u and q by constant values, i.e. $u_k := u(\boldsymbol{x}_k)$ and $q_k := q(\boldsymbol{x}_k)$. The discretization transforms the boundary integral equation (2) into

$$\tilde{\mathbf{H}}\mathbf{u} = \mathbf{G}\mathbf{q}. \tag{4}$$

eigenvalues $\mathcal{K}^s$	eigenfunctions $\mathcal{K}^s$
$-R\log R$	1
$\frac{R}{2n}$	$\sin(n\theta)$
	$\cos(n\theta)$

eigenvalues $\mathcal{K}^d$	eigenfunctions $\mathcal{K}^d$
$-\frac{1}{2}$	1
0	$\sin(n\theta)$
	$\cos(n\theta)$

Table 1: *The eigenvalues and eigenfunctions of $\mathcal{K}^s$ and $\mathcal{K}^d$.*

Here $\tilde{\mathbf{H}} := \frac{1}{2}\mathbf{I} + \mathbf{H}$ and the matrix elements of $\mathbf{H}$ and $\mathbf{G}$ are given by

$$\begin{aligned} G_{lk} &:= -\frac{1}{2\pi}\int_{\Gamma_k} \ln\|\boldsymbol{x}_l - \boldsymbol{x}_Q\|\Gamma_Q, \\ H_{lk} &:= \frac{1}{2\pi}\int_{\Gamma_k} \frac{< \boldsymbol{x}_l - \boldsymbol{x}_Q, \boldsymbol{n}_Q >}{\|\boldsymbol{x}_l - \boldsymbol{x}_Q\|^2} d\Gamma_Q. \end{aligned} \tag{5}$$

The matrix $\mathbf{I}$ is the $N \times N$ identity matrix and the vectors $\mathbf{u} := [u_1, \ldots, u_N]^T$ and $\mathbf{q} := [q_1, \ldots, q_N]^T$ are the coefficient vectors of u and q.

The first m coefficients of $\mathbf{u}$ are given by the Dirichlet conditions and the last $N - m$ coefficients of $\mathbf{q}$ are given by the Neuman conditions. To obtain a standard linear system with all the unknowns on the left-hand side, we need to move the first m columns from the left-hand side in (4) to the right, and replace them by the first m columns from the right-hand side. To this end we introduce the projection matrices

$$\mathbf{P}_1 := \left[\frac{\mathbf{I}_m}{\emptyset}\right], \qquad \mathbf{P}_2 := \left[\frac{\emptyset}{\mathbf{I}_{N-m}}\right], \tag{6}$$

where $\mathbf{I}_k$ is the identity matrix with size k. The system in (4) is written as

$$\mathbf{A}\mathbf{x} = \tilde{\mathbf{G}}\mathbf{b}, \tag{7}$$

with the matrices $\mathbf{A}$ and $\tilde{\mathbf{G}}$ given by

$$\begin{aligned} \mathbf{A} &:= [-\mathbf{G}\mathbf{P}_1 \mid \tilde{\mathbf{H}}\mathbf{P}_2], \\ \tilde{\mathbf{G}} &:= [-\tilde{\mathbf{H}}\mathbf{P}_1 \mid \mathbf{G}\mathbf{P}_2], \end{aligned} \tag{8}$$

and the vectors $\mathbf{x}$ and $\mathbf{b}$ by

$$\begin{aligned} \mathbf{x} &:= [q_1, \ldots, q_m, u_{m+1}, \ldots, u_N]^T, \\ \mathbf{b} &:= [u_1, \ldots, u_m, q_{m+1}, \ldots, q_N]^T. \end{aligned} \tag{9}$$

The system matrix $\mathbf{A}$ in (7) and the right-hand side $\tilde{\mathbf{G}}\mathbf{b}$ are known. All unknown coefficients are in the vector $\mathbf{x}$. In the remainder of this paper we focus on estimating the condition number of the matrix $\mathbf{A}$.

By introducing polar coordinates we can calculate the matrix elements H_{lk} in (5) explicitly. Then it can be shown that all diagonal elements of $\tilde{\mathbf{H}}$ have the value $1/2$, while

all off-diagonal elements have the value $-1/2\pi \tan(\pi/N)$. For this *circulant* matrix [13] it is possible to give the exact discrete eigenvalues, see [3]. We find

$$\begin{aligned} \lambda_1(\tilde{\mathbf{H}}) &= \frac{1}{2} + \frac{1}{2\pi} \tan \frac{\pi}{N}, \text{ (mult. } N-1), \\ \lambda_N(\tilde{\mathbf{H}}) &= \frac{1}{2} - \frac{N-1}{2\pi} \tan \frac{\pi}{N}, \text{ (mult. } 1). \end{aligned} \tag{10}$$

The matrices $\mathbf{G}$ and $\mathbf{H}$ directly relate to the integral operators $\mathcal{K}^s$ and $\mathcal{K}^d$. In the limit case $N \to \infty$ the discrete eigenvalues of $\mathbf{G}$ and $\mathbf{H}$ are equal to the analytical eigenvalues of $\mathcal{K}^s$ and $\mathcal{K}^d$. Hence, in the remainder of this paper we estimate the eigenvalues of $\mathbf{G}$ by the eigenvalues of $\mathcal{K}^s$. For $\tilde{\mathbf{H}}$ we use the eigenvalues in (10).

3 Estimating the condition number of $\mathbf{A}$

In this section we show that the matrix $\mathbf{A}$ can be decomposed into four matrices; a unitary matrix $\mathbf{F}$, a diagonal matrix $\mathbf{D}$, a matrix $\mathbf{Q}$ that consists of two blocks with orthogonal columns, and a upper triangular matrix $\mathbf{U}$. For the latter three matrices we derive estimates for the condition number.

3.1 Decomposition

Due to the symmetry of the discretization of the boundary the matrices $\mathbf{G}$ and $\tilde{\mathbf{H}}$ are circulant matrices (see [3]) and can therefore be decomposed as

$$\begin{aligned} \mathbf{G} &= \mathbf{F}^* \mathbf{\Lambda}_G \mathbf{F}, \\ \tilde{\mathbf{H}} &= \mathbf{F}^* \mathbf{\Lambda}_H \mathbf{F}. \end{aligned} \tag{11}$$

Here $\mathbf{\Lambda}_G$ and $\mathbf{\Lambda}_H$ are diagonal matrices containing the eigenvalues of $\mathbf{G}$ and $\tilde{\mathbf{H}}$ respectively. The matrix $\mathbf{F}$ is the so called *Fourier matrix*, whose elements are defined by

$$F^*_{ij} := \frac{1}{\sqrt{N}} w^{(i-1)(j-1)}, \tag{12}$$

where the asterisk denotes complex conjugate transpose and $w := e^{2\pi i/N}$ is the N-th root of unity. Substituting the decompositions for $\mathbf{G}$ and $\tilde{\mathbf{H}}$, we write $\mathbf{A}$ in (8) as

$$\mathbf{A} = \mathbf{F}^* [-\mathbf{\Lambda}_G \mathbf{F} \mathbf{P}_1 \mid \mathbf{\Lambda}_H \mathbf{F} \mathbf{P}_2]. \tag{13}$$

Note that the eigenvalues of $\mathbf{\Lambda}_G$ are known via the eigenvalues of the corresponding integral operator and the eigenvalues of $\mathbf{\Lambda}_H$ are given in (10). We define $\mathbf{F}_1 := \mathbf{F}\mathbf{P}_1$ and $\mathbf{F}_2 := \mathbf{F}\mathbf{P}_2$ to write

$$\mathbf{A} = \mathbf{F}^* [-\mathbf{\Lambda}_G \mathbf{F}_1 \mid \mathbf{\Lambda}_H \mathbf{F}_2]. \tag{14}$$

By introducing two other diagonal matrices $\mathbf{\Lambda}$ and $\mathbf{D}$ according to $\mathbf{\Lambda} := \mathbf{\Lambda}_G^{1/2} \mathbf{\Lambda}_H^{-1/2}$ and $\mathbf{D} := \mathbf{\Lambda}_G^{1/2} \mathbf{\Lambda}_H^{1/2}$ we obtain

$$\mathbf{A} = \mathbf{F}^* \mathbf{D} [-\mathbf{\Lambda} \mathbf{F}_1 \mid \mathbf{\Lambda}^{-1} \mathbf{F}_2]. \tag{15}$$

We also introduce QU-decompositions of $\mathbf{\Lambda F}_1$ and $\mathbf{\Lambda}^{-1}\mathbf{F}_2$ by

$$\begin{aligned} \mathbf{\Lambda F}_1 &= \mathbf{Q}_1\mathbf{U}_1, \\ \mathbf{\Lambda}^{-1}\mathbf{F}_2 &= \mathbf{Q}_2\mathbf{U}_2. \end{aligned} \tag{16}$$

The matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ have size $N \times m$ and $N \times (N-m)$ and are an orthogonal basis of the subspaces which are spanned by the columns of $\mathbf{\Lambda F}_1$ and $\mathbf{\Lambda}^{-1}\mathbf{F}_2$ respectively. The matrices $\mathbf{U}_1$ and $\mathbf{U}_2$ have size $m \times m$ and $(N-m) \times (N-m)$ and are upper triangular matrices. With these decompositions $\mathbf{A}$ is written as

$$\mathbf{A} = \mathbf{F}^*\mathbf{D} \underbrace{\left[-\mathbf{Q}_1 \mid \mathbf{Q}_2\right]}_{\mathbf{Q}} \underbrace{\begin{bmatrix} \mathbf{U}_1 & \emptyset \\ \emptyset & \mathbf{U}_2 \end{bmatrix}}_{\mathbf{U}} = \mathbf{F}^*\mathbf{DQU}. \tag{17}$$

Since the unitary matrix $\mathbf{F}$ has condition number one we find

$$\mathrm{cond}(\mathbf{A}) \leq \mathrm{cond}(\mathbf{D})\,\mathrm{cond}(\mathbf{Q})\,\mathrm{cond}(\mathbf{U}). \tag{18}$$

3.2 Estimates of the condition numbers

The matrix $\mathbf{D} = \mathbf{\Lambda}_G^{1/2}\mathbf{\Lambda}_H^{1/2}$ is the product of two diagonal matrices. Hence the eigenvalues of $\mathbf{D}$ are the products of the eigenvalues of $\mathbf{\Lambda}_G^{1/2}$ and $\mathbf{\Lambda}_H^{1/2}$. We estimate the eigenvalues of $\mathbf{\Lambda}_G$ by the eigenvalues of $\mathcal{K}^s$ in Table 1 and we use the exact eigenvalues as given in (10). We find

$$\begin{aligned} \sigma_1(\mathbf{D})^2 &= \max\left(R|\log R|\left(\frac{1}{2} - \frac{N-1}{2\pi}\tan\frac{\pi}{N}\right), \frac{R}{2}\left(\frac{1}{2} + \frac{1}{2\pi}\tan\frac{\pi}{N}\right)\right), \\ \sigma_N(\mathbf{D})^2 &= \min\left(R|\log R|\left(\frac{1}{2} - \frac{N-1}{2\pi}\tan\frac{\pi}{N}\right), \frac{R}{N}\left(\frac{1}{2} + \frac{1}{2\pi}\tan\frac{\pi}{N}\right)\right) \end{aligned} \tag{19}$$

from which the condition number of $\mathbf{D}$ is easily obtained.

The condition number of $\mathbf{Q}$ is related to the Kantorovich-Wielandt angle θ (see [14]) according to

$$\mathrm{cond}(\mathbf{Q}) = \frac{1}{\tan(\theta/2)}, \tag{20}$$

where θ is defined as

$$\cos\theta := \max_{\mathbf{x}\perp\mathbf{y}} \frac{|(\mathbf{Qx}, \mathbf{Qy})|}{\|\mathbf{Qx}\|\|\mathbf{Qy}\|}. \tag{21}$$

In Section 3.3 we show that the angle θ is also the angle between the subspaces that are spanned by the columns of $\mathbf{\Lambda F}_1$ and $\mathbf{\Lambda}^{-1}\mathbf{F}_2$. We cannot say *a priori* whether the angle between the subspaces is small or close to $\pi/2$. In the following theorem we show that in our case the two subspaces are perpendicular.

Theorem 1 *For the angle θ as defined above, we have* $\cos\theta = 0$.

Proof. Since $\mathbf{Q} = [-\mathbf{Q}_1 \mid \mathbf{Q}_2]$ we can split two orthogonal vectors $\mathbf{x}$ and $\mathbf{y}$ into $\mathbf{x} = [\mathbf{x}_1^T, \mathbf{x}_2^T]^T$ and $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T$, such that

$$\begin{aligned}(\mathbf{Q}\mathbf{x}, \mathbf{Q}\mathbf{y}) &= (\mathbf{Q}_1\mathbf{x}_1, \mathbf{Q}_1\mathbf{y}_1) + (\mathbf{Q}_2\mathbf{x}_2, \mathbf{Q}_2\mathbf{y}_2) \\ &\quad -(\mathbf{Q}_1\mathbf{x}_1, \mathbf{Q}_2\mathbf{y}_2) - (\mathbf{Q}_2\mathbf{x}_2, \mathbf{Q}_1\mathbf{y}_1). \end{aligned} \tag{22}$$

The matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are orthogonal, which implies that $(\mathbf{Q}_1\mathbf{x}_1, \mathbf{Q}_1\mathbf{y}_1) = (\mathbf{x}_1, \mathbf{y}_1)$ and $(\mathbf{Q}_2\mathbf{x}_2, \mathbf{Q}_2\mathbf{y}_2) = (\mathbf{x}_2, \mathbf{y}_2)$. Hence the inner product becomes

$$(\mathbf{Q}\mathbf{x}, \mathbf{Q}\mathbf{y}) = (\mathbf{x}_1, \mathbf{y}_1) + (\mathbf{x}_2, \mathbf{y}_2) - (\mathbf{Q}_1\mathbf{x}_1, \mathbf{Q}_2\mathbf{y}_2) - (\mathbf{Q}_2\mathbf{x}_2, \mathbf{Q}_1\mathbf{y}_1). \tag{23}$$

Because $\mathbf{x}$ and $\mathbf{y}$ are perpendicular we have $(\mathbf{x}, \mathbf{y}) = (\mathbf{x}_1, \mathbf{y}_1) + (\mathbf{x}_2, \mathbf{y}_2) = 0$. Thus we obtain

$$(\mathbf{Q}\mathbf{x}, \mathbf{Q}\mathbf{y}) = -(\mathbf{Q}_1\mathbf{x}_1, \mathbf{Q}_2\mathbf{y}_2) - (\mathbf{Q}_2\mathbf{x}_2, \mathbf{Q}_1\mathbf{y}_1). \tag{24}$$

Consider the first inner product on the right-hand side. Recall that $\mathbf{Q}_1$ is a basis for the column space of $\mathbf{\Lambda F}_1$ and $\mathbf{Q}_2$ is a basis for the column space of $\mathbf{\Lambda}^{-1}\mathbf{F}_2$. Therefore we can write for $\boldsymbol{\xi}_1 \in \mathbb{R}^m$ and $\boldsymbol{\xi}_2 \in \mathbb{R}^{N-m}$

$$\begin{aligned}\mathbf{Q}_1\mathbf{x}_1 &= \mathbf{\Lambda F}_1\boldsymbol{\xi}_1 =: \mathbf{\Lambda}\boldsymbol{\eta}_1,\ \boldsymbol{\eta}_1 \in \mathcal{R}(\mathbf{F}_1), \\ \mathbf{Q}_2\mathbf{y}_2 &= \mathbf{\Lambda}^{-1}\mathbf{F}_2\boldsymbol{\xi}_2 =: \mathbf{\Lambda}^{-1}\boldsymbol{\eta}_2,\ \boldsymbol{\eta}_2 \in \mathcal{R}(\mathbf{F}_2). \end{aligned} \tag{25}$$

The inner product of $\mathbf{Q}_1\mathbf{x}_1$ and $\mathbf{Q}_2\mathbf{y}_2$ yields

$$(\mathbf{Q}_1\mathbf{x}_1, \mathbf{Q}_2\mathbf{y}_2) = (\mathbf{\Lambda}\boldsymbol{\eta}_1, \mathbf{\Lambda}^{-1}\boldsymbol{\eta}_2) = (\boldsymbol{\eta}_1, \boldsymbol{\eta}_2) = (\mathbf{F}_1\boldsymbol{\xi_1}, \mathbf{F}_2\boldsymbol{\xi_2}) = (\mathbf{F}_2^T\mathbf{F}_1\boldsymbol{\xi_1}, \boldsymbol{\xi_2}) = 0, \tag{26}$$

since $\mathcal{R}(\mathbf{F}_1) \perp \mathcal{R}(\mathbf{F}_2)$. Likewise, for the second inner product on the right-hand side of (24) we find that $(\mathbf{Q}_2\mathbf{x}_2, \mathbf{Q}_1\mathbf{y}_1) = 0$. As a consequence $(\mathbf{Q}\mathbf{x}, \mathbf{Q}\mathbf{y}) = 0$ and with (21) $\cos\theta = 0$. ■

Together with (20), Theorem 1 implies that

$$\mathrm{cond}(\mathbf{Q}) = 1. \tag{27}$$

To estimate the condition number of $\mathbf{U}$ we will estimate the singular values of $\mathbf{U}_1$ and $\mathbf{U}_2$. We observe that

$$\begin{aligned}\sigma_k(\mathbf{U}_1) &= \sigma_k(\mathbf{Q}_1\mathbf{U}_1) = \sigma_k(\mathbf{\Lambda F}_1) \le \sigma_k(\mathbf{\Lambda})\sigma_1(\mathbf{F}_1) \\ &= \sigma_k(\mathbf{\Lambda}),\ k = 1, \ldots, m, \\ \sigma_k(\mathbf{U}_2) &= \sigma_k(\mathbf{Q}_2\mathbf{U}_2) = \sigma_k(\mathbf{\Lambda}^{-1}\mathbf{F}_2) \le \sigma_k(\mathbf{\Lambda}^{-1})\ \sigma_1(\mathbf{F}_2) \\ &= \sigma_k(\mathbf{\Lambda}^{-1}),\ k = 1, \ldots, N-m, \end{aligned} \tag{28}$$

where we used the facts that $\mathbf{Q}_i$ and $\mathbf{F}_i$ have orthogonal columns and singular values 1. The estimates for singular values of product matrices follow from [15, p. 30]. Furthermore, with $\mathbf{F}_1 = \mathbf{\Lambda}^{-1}\mathbf{Q}_1\mathbf{U}_1$ and $\mathbf{F}_2 = \mathbf{\Lambda}\mathbf{Q}_2\mathbf{U}_2$ we obtain

$$\begin{aligned}1 &= \sigma_k(\mathbf{F}_1) = \sigma_k(\mathbf{\Lambda}^{-1}\mathbf{Q}_1\mathbf{U}_1) \le \sigma_1(\mathbf{\Lambda}^{-1})\sigma_k(\mathbf{Q}_1\mathbf{U}_1) \\ &= \sigma_1(\mathbf{\Lambda}^{-1})\sigma_k(\mathbf{U}_1),\ k = 1, \ldots, m, \\ 1 &= \sigma_k(\mathbf{F}_2) = \sigma_k(\mathbf{\Lambda}\mathbf{Q}_2\mathbf{U}_2) \le \sigma_1(\mathbf{\Lambda})\sigma_k(\mathbf{Q}_2\mathbf{U}_2) \\ &= \sigma_1(\mathbf{\Lambda})\sigma_k(\mathbf{U}_2),\ k = 1, \ldots, N-m, \end{aligned} \tag{29}$$

which yields

$$\begin{aligned}\sigma_k(\mathbf{U}_1) &\geq \frac{1}{\sigma_1(\mathbf{\Lambda}^{-1})} = \sigma_N(\mathbf{\Lambda}),\ k = 1, \ldots, m,\\ \sigma_k(\mathbf{U}_2) &\geq \frac{1}{\sigma_1(\mathbf{\Lambda})} = \sigma_N(\mathbf{\Lambda}^{-1}),\ k = 1, \ldots, N-m. \end{aligned} \tag{30}$$

With (28) and (30) we have upper and lower bounds for the singular values of $\mathbf{U}_1$ and $\mathbf{U}_2$. The set of singular values of $\mathbf{U}$ is composed of the singular values of $\mathbf{U}_1$ and the singular values of $\mathbf{U}_2$. For the condition number of $\mathbf{U}$ we obtain

$$\begin{aligned}\mathrm{cond}(\mathbf{U}) &= \frac{\max\big(\sigma_1(\mathbf{U}_1), \sigma_1(\mathbf{U}_1)\big)}{\min\big(\sigma_N(\mathbf{U}_1), \sigma_N(\mathbf{U}_1)\big)} \leq \frac{\max(\sigma_1(\mathbf{\Lambda}), \sigma_1(\mathbf{\Lambda}^{-1}))}{\min(\sigma_N(\mathbf{\Lambda}), \sigma_N(\mathbf{\Lambda}^{-1}))}\\ &= \frac{\max\left(\sigma_1(\mathbf{\Lambda}), \dfrac{1}{\sigma_N(\mathbf{\Lambda})}\right)}{\min\left(\sigma_N(\mathbf{\Lambda}), \dfrac{1}{\sigma_1(\mathbf{\Lambda})}\right)} = \max\left(\sigma_1(\mathbf{\Lambda})^2, \frac{1}{\sigma_N(\mathbf{\Lambda})^2}\right). \end{aligned} \tag{31}$$

Since $\mathbf{\Lambda}$ is the product of the square roots of $\mathbf{\Lambda}_G$ and $\mathbf{\Lambda}_H^{-1}$, we can derive its singular values, namely

$$\begin{aligned}\sigma_1(\mathbf{\Lambda})^2 &= \max\left(\frac{2\pi R|\log R|}{\pi - (N-1)\tan\pi/N}, \frac{\pi R}{\pi + \tan\pi/N}\right),\\ \sigma_N(\mathbf{\Lambda})^2 &= \min\left(\frac{2\pi R|\log R|}{\pi - (N-1)\tan\pi/N}, \frac{2\pi R/N}{\pi + \tan\pi/N}\right). \end{aligned} \tag{32}$$

3.3 Angle between subspaces

The angle α between the two subspaces that are spanned by the columns of $\mathbf{F}_1\mathbf{\Lambda}$ and $\mathbf{F}_2\mathbf{\Lambda}^{-1}$ is defined as [16]

$$\cos\alpha := \max_{\boldsymbol{\xi}_1 \in \mathcal{R}(\mathbf{F}_1\mathbf{\Lambda})} \max_{\boldsymbol{\xi}_2 \in \mathcal{R}(\mathbf{F}_2\mathbf{\Lambda}^{-1})} \frac{|(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)|}{\|\boldsymbol{\xi}_1\|\|\boldsymbol{\xi}_2\|}. \tag{33}$$

We will show that this angle α is equal to the Kantorovich-Wielandt angle θ defined in (21). Recall that the angle θ is found by taking pairs of orthogonal vectors $\mathbf{x}$ and $\mathbf{y}$ and calculating the smallest angle between their images under $\mathbf{Q}$. The matrix $\mathbf{Q}$ consists of two blocks, and therefore we select two special vectors $\mathbf{x}$ and $\mathbf{y}$, namely $\mathbf{x}^T = [\mathbf{x}_1^T \mid 0, \ldots, 0]^T$ and $\mathbf{y}^T = [0, \ldots, 0 \mid \mathbf{y}_1^T]^T$, where $\mathbf{x}_1 \in \mathbb{R}^m$ and $\mathbf{y}_1 \in \mathbb{R}^{N-m}$. Clearly we have $\mathbf{x} \perp \mathbf{y}$. Moreover, we observe that $\mathbf{Q}\mathbf{x} = -\mathbf{Q}_1\mathbf{x}_1$ and $\mathbf{Q}\mathbf{y} = \mathbf{Q}_2\mathbf{y}_1$. We substitute this into the definition of the Kantorovich-Wielandt angle and find

$$\cos\theta = \max_{\mathbf{x}_1 \in \mathbb{R}^m} \max_{\mathbf{y}_1 \in \mathbb{R}^{N-m}} \frac{|(\mathbf{Q}_1\mathbf{x}_1, \mathbf{Q}_2\mathbf{y}_1)|}{\|\mathbf{Q}_1\mathbf{x}_1\|\|\mathbf{Q}_2\mathbf{y}_1\|}. \tag{34}$$

Recall that the matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are an orthogonal basis for the subspaces spanned by the columns of $\mathbf{F}_1\mathbf{\Lambda}$ and $\mathbf{F}_2\mathbf{\Lambda}^{-1}$. This means that we can introduce $\boldsymbol{\xi}_1 \in \mathcal{R}(\mathbf{F}_1\mathbf{\Lambda})$ and

$\boldsymbol{\xi}_2 \in \mathcal{R}(\mathbf{F}_2\boldsymbol{\Lambda}^{-1})$ such that $\boldsymbol{\xi}_1 = \mathbf{Q}_1\mathbf{x}_1$ and $\boldsymbol{\xi}_2 = \mathbf{Q}_2\mathbf{y}_1$. Then (34) becomes

$$\cos\theta = \max_{\boldsymbol{\xi}_1 \in \mathcal{R}(\mathbf{F}_1\boldsymbol{\Lambda})} \max_{\boldsymbol{\xi}_2 \in \mathcal{R}(\mathbf{F}_2\boldsymbol{\Lambda}^{-1})} \frac{|(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)|}{\|\boldsymbol{\xi}_1\|\|\boldsymbol{\xi}_2\|}. \tag{35}$$

This is the definition of the angle between the subspaces. Thus the Kantorovich-Wielandt angle is equal to the angle between the two subspaces.

4 Applications to the problem matrix

We demonstrate the estimate of the condition number of the BEM system matrix for several cases. First we look at the two extreme cases: the BVP with purely Dirichlet conditions ($m = N$) and the BVP with purely Neuman conditions ($m = 0$). Then we deal with the case in which the boundary BVP has both Dirichlet and Neuman conditions ($0 < m < N$).

4.1 Dirichlet conditions

In the situation $m = N$ we only have Dirichlet conditions on the boundary. In this case $\mathbf{A} = -\mathbf{G}$. The singular values of $\mathbf{G}$ are approximated by the eigenvalues of $\mathcal{K}^s$ and we easily obtain an estimate for the condition number of $\mathbf{A}$,

$$\mathrm{cond}(\mathbf{A}) \approx \frac{\max(\frac{1}{2}, |\log R|)}{\min(\frac{1}{N}, |\log R|)}. \tag{36}$$

Figure 2 shows the graph of the condition number of $\mathbf{A}$ as a function of radius R. We perform calculations for $N = 8$ (red), $N = 12$ (blue), $N = 16$ (black), and $N = 20$ (green), cf [2].

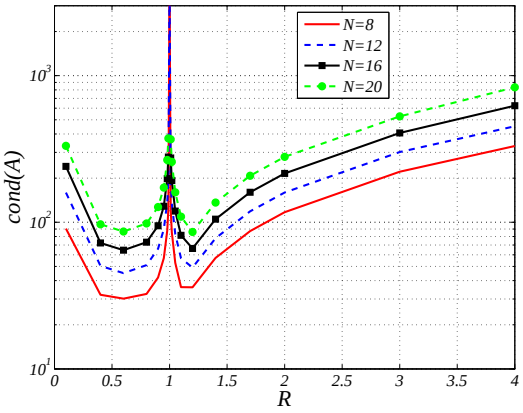

Figure 2: *Condition number of* $\mathbf{A}$ *as a function of* R *when* $m = N$ *(Dirichlet problem). Four values of* N *are plotted:* $N = 8$ *(red),* $N = 12$ *(blue),* $N = 16$ *(black) and* $N = 20$ *(green).*

We observe that for increasing N the condition number also increases. For $R = 1$ the condition number becomes infinitely large, which is caused by the factor $|\log R|$ in the denominator in expression (36). This singularity is interesting since it implies that we cannot cope with circular domains for which the radius equals one. Such a situation often occurs if we take dimensionless spatial coordinates.

4.2 Neuman conditions

If $m = 0$ we only have Neuman conditions on the boundary. In this case $\mathbf{A} = \tilde{\mathbf{H}}$. Since all the eigenvalues of $\tilde{\mathbf{H}}$ are known, we easily determine the condition number of $\mathbf{A}$,

$$\mathrm{cond}(\mathbf{A}) = \frac{\pi + \tan\frac{\pi}{N}}{|\pi - (N-1)\tan\frac{\pi}{N}|}. \tag{37}$$

Note that the parameter R is not present in this expression. Therefore the condition number of $\mathbf{A}$ is independent of R. Figure 3 shows the condition number of $\mathbf{A}$ as a function of N. We see that the condition number is asymptotically linearly dependent on N. In the neighborhood of $N = 4$ we observe a local maximum in the condition number. This is caused by the denominator in expression (37), which becomes zero between $N = 3$ and $N = 4$. However, in practical applications N will always be chosen larger than 4.

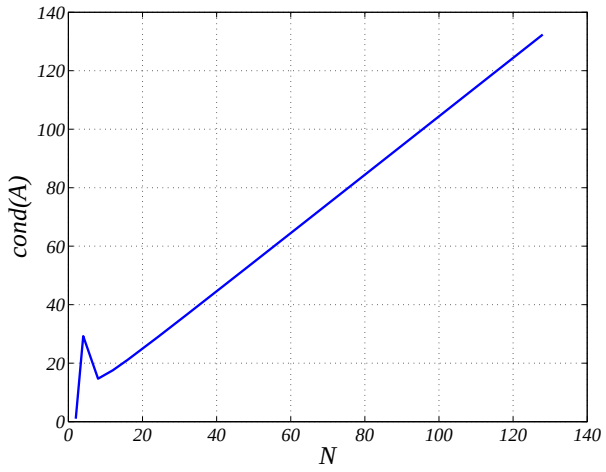

Figure 3: *Condition number of* $\mathbf{A}$ *as a function of* N *when* $m = 0$ *(Neuman problem).*

4.3 Mixed boundary conditions

For ease of notation let the number of boundary elements N be even. We choose $m = N/2$, which means that there are as many elements with Dirichlet conditions as there are elements with Neuman conditions. We use a first order approximation for $\tan \pi/N$ to

approximate the largest and smallest singular values of the matrices $\mathbf{D}$ and $\mathbf{\Lambda}$ as given in (19) and (32). We find

$$\begin{aligned}
\sigma_1(\mathbf{D})^2 &\approx \frac{R}{4N}\max\big(2|\log R|, N+1\big),\\
\sigma_N(\mathbf{D})^2 &\approx \frac{R}{2N}\min\Big(|\log R|, 1+\frac{1}{N}\Big),\\
\sigma_1(\mathbf{\Lambda})^2 &\approx RN\max\Big(2|\log R|, \frac{1}{N+1}\Big),\\
\sigma_N(\mathbf{\Lambda})^2 &\approx 2RN\min\Big(|\log R|, \frac{1}{N(N+1)}\Big).
\end{aligned} \tag{38}$$

Choosing $R = 1/2$ we get

$$\begin{aligned}
\mathrm{cond}(\mathbf{D}) &\approx \sqrt{\frac{N+1}{2\log 2}},\\
\mathrm{cond}(\mathbf{U}) &\leq N+1,
\end{aligned} \tag{39}$$

which gives the following estimate for the condition number of $\mathbf{A}$,

$$\mathrm{cond}(\mathbf{A}) \leq \frac{1}{\sqrt{2\log 2}}(N+1)^{3/2}. \tag{40}$$

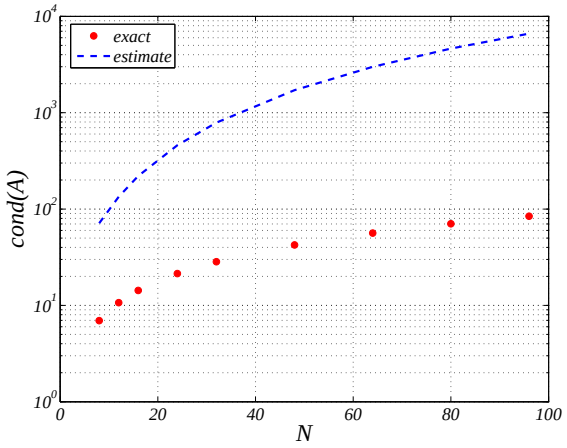

Figure 4: *Condition number of* $\mathbf{A}$ *on a logarithmic scale as a function of* N *with* $R = 1/2$ *and* $m = N/2$*. The blue line is the estimate whereas the red dots give the exact condition number for several values of* N*.*

In Figure 4 we show this estimate for $\mathrm{cond}(\mathbf{A})$ on a logarithmic scale. The blue line gives the estimate as a function of N, whereas the red dots gives the exact value of $\mathrm{cond}(\mathbf{A})$ for several values of N. Thus we obtain an upper bound for the condition number of the

system matrix. Note that the data shows a linear behavior in N while our estimate is of order $N^{3/2}$. At a later point in this paper we elaborate on this difference and its cause.

Figure 5 gives the dependency of the condition number on R and its estimate. We typically choose $N = 12$ and $m = 6$, i.e. again as many elements with Dirichlet conditions as elements with Neuman conditions. Again we obtain an upper bound for the condition number of $\mathbf{A}$.

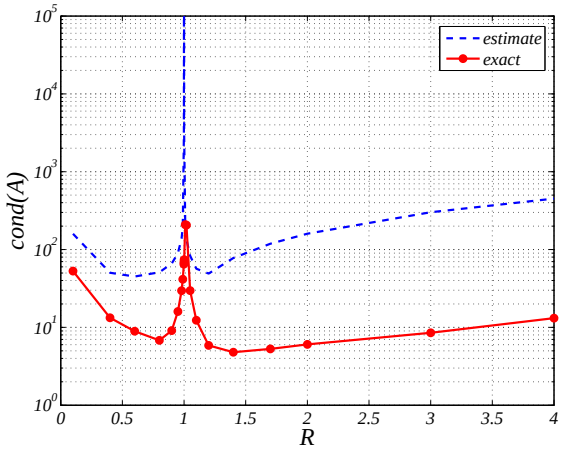

Figure 5: *Condition number of* $\mathbf{A}$ *as a function of* R *with* $N = 12$ *and* $m = 6$*. The blue line is the estimate whereas the red line gives exact values of the condition number for several values of* R*.*

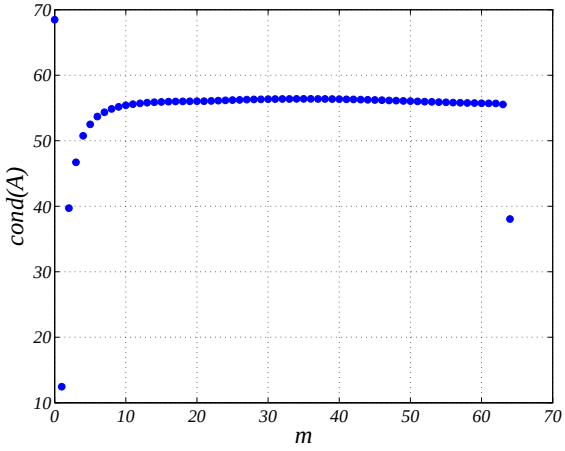

Figure 6: *Condition number of* $\mathbf{A}$ *as a function of* m *with* $N = 64$ *and* $R = 1/2$*.*

The expressions for the singular values of $\mathbf{D}$ and $\mathbf{\Lambda}$ do not contain the parameter m. Therefore the estimate for the condition number $\mathbf{A}$ is independent of m. The actual condition number does depend on m. In other words our estimate is an upper bound for

all possible choices of m. Figure 6 shows the condition number as a function of m. We choose $R = 1/2$, $N = 64$, and m ranges from 0 to N. We observe that for $m = 0$, i.e. the Neuman problem, the condition number is largest. The lowest condition number is obtained when $m = 1$, which is the situation in which one element has a Dirichlet condition and all the other elements have Neuman conditions. For a large range of values of m, $10 < m < N$, the condition numbers are comparable. For $m = N$, the Dirichlet problem, the condition number is considerably lower compared to $m = N - 1$.

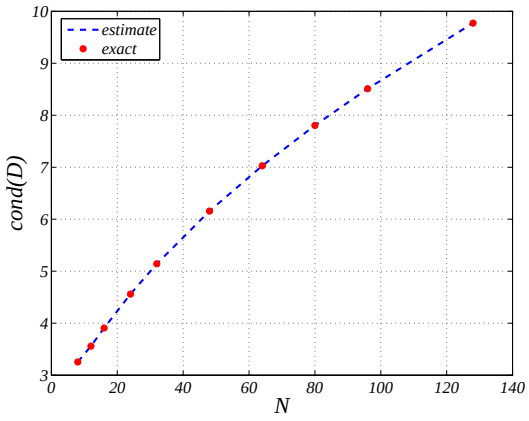

Figure 7: *Condition number of* $\mathbf{D}$ *as a function of* N *with* $R = 1/2$ *and* $m = N/2$*. The blue line is the estimate whereas the red dots give the exact value for several values of* N*.*

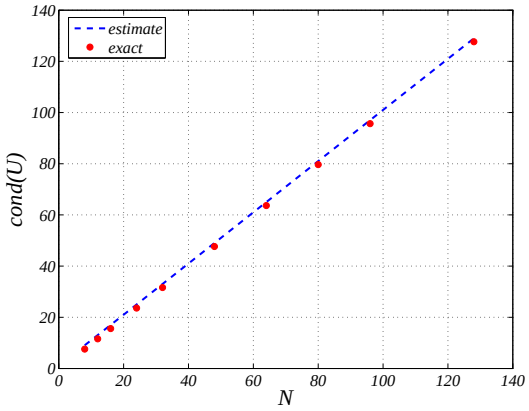

Figure 8: *Condition number of* $\mathbf{U}$ *as a function of* N*, with* $R = 1/2$ *and* $m = N/2$*. The blue line is the estimate whereas the red dots give the exact value for several values of* N*.*

Figure 7 shows the condition number of $\mathbf{D}$ as a function of N. The red dots represent the exact condition number for several values of N. The blue line gives the approximation as given in (39). We observe that the condition number of $\mathbf{D}$ is approximated very well.

Hence the difference that we observed between estimate and exact value of $\mathrm{cond}(\mathbf{A})$ is not caused by a poor estimate of $\mathrm{cond}(\mathbf{D})$.

We also plot the condition number of $\mathbf{U}$ and its approximation in Figure 8. Again we observe a very good correspondence between the exact value of $\mathrm{cond}(\mathbf{U})$ and the approximation. Hence the difference between the estimate and the exact value of $\mathrm{cond}(\mathbf{A})$ is not caused by a poor estimate of $\mathrm{cond}(\mathbf{U})$.

To recapitulate, we decompose the matrix $\mathbf{A}$ into a product of four matrices and derive that $\mathrm{cond}(\mathbf{A}) \leq \mathrm{cond}(\mathbf{D})\mathrm{cond}(\mathbf{U})$. We approximate the condition numbers of $\mathbf{D}$ and $\mathbf{U}$ very well. Unfortunately the estimate of the condition number of $\mathbf{A}$ is too large. Clearly the reason for this is that

$$\|\mathbf{DU}\| \leq \|\mathbf{D}\|\|\mathbf{U}\|. \tag{41}$$

This occurs in (18), where we approximate the condition number of $\mathbf{A}$ by the product of the condition numbers of $\mathbf{D}$, $\mathbf{Q}$ and $\mathbf{U}$.

5 Conclusion

The previous section shows that we find an upper bound for the condition number of the system matrix $\mathbf{A}$. It demonstrates that when $R = 1/2$ the condition number is at most $N^{3/2}$, indicating that the matrix is well-conditioned. The same is true if R is not too close to one. Thus, the boundary element method applied to the Laplace equation on a circle with mixed boundary conditions leads to a well-conditioned linear system of algebraic equations. It is not *a priori* clear whether the system matrix will be well-conditioned. The matrix is constructed from two blocks, the columns of each block representing a subspace. In principle, nothing guarantees us that these two subspaces are orthogonal. In fact, when the angle between these subspaces is very small, the condition number of the system matrix is very large. However, we prove that the two subspaces are perpendicular, leading to a well-conditioned matrix. We restrict ourselves to a simple case, but our research could be extended to more complex geometries, for instance a square or rectangular geometry.

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