Regularization of the Divergent Integrals II. Application in Fracture Mechanics

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Abstract

In this article the methodology for divergent integral regularization developed in [9] is applied for the regularization of the weakly singular and hypersingular integrals, which arise when the boundary integral equation (BIE) methods are used to solve fracture mechanics problems. The approach is based on the theory of distribution and the application of the Green theorem. The weakly singular and hypersingular integrals over arbitrary convex polygon are transformed into the regular contour integrals that can be easily calculated analytically or numerically.

Key words: weakly singular, singular, hypersingular integrals, boundary integral equations, fracture mechanics

1. Introduction

In [6] it was shown that the divergent integrals and the integral operators with divergent kernels are the main difficulty when problems in mathematics, applied science and engineering are solved by the BIE methods. Different methods have been developed for regularization of weakly singular (*WS*), singular, and hypersingular integrals. The weakly singular and singular integrals and integral operators with such kernels have a well-established theoretical basis [5, 6]. The hypersingular integrals had been considered by Hadamard in the sense of finite part (*FP*) in [4]. The theory of distributions allows us to study the divergent integrals and integral operators with kernels containing different kind of singularities in the same way as the regular integrals.

We applied the theory of distribution approach for the first time in [8], which was further developed in [10-12] and applied for fracture dynamics problems in our numerous publications [1-3]. In our earlier publications the regularization technique was based on Green theorem and for each divergent integral specific method had to be developed. In contrast, equations presented in [9] can be applied for a wide class of divergent integral regularizations universally. In the present paper these equations are applied for the regularization of the weakly singular and hypersingular integrals that arise in the BIE methods used to solve fracture mechanics problems.

2. Boundary integral equations

Let a three-dimensional linearly elastic homogeneous isotropic space contain a plane crack with a surface S. Let us introduce Cartesian coordinates system, with x_1 and x_2 axes in the plane of the crack, and the x_3 axis perpendicular to this plane. In [7, 10] it was shown that the BIE that relate load $p_i(\mathbf{y})$ on the crack faces and crack opening $\Delta u_j(\mathbf{x})$ may be written in the following form

$$p_i^k(\mathbf{y}) = -\int_S F_{ij}(\mathbf{x}-\mathbf{y},k)\Delta u_j^k(\mathbf{x})\,d\Omega \ . \tag{1}$$

Here k is the Laplace transform parameter for the Laplace transform domain and is the wave frequency for the frequency domain formulations, which is zero for the static problems. The kernels $F_{ij}(\mathbf{x}-\mathbf{y},k)$ in the BIE (1) may be presented in the form [3]

$$F_{13}(\mathbf{x},\mathbf{y},\omega_k) = F_{31}(\mathbf{x},\mathbf{y},\omega_k) = F_{23}(\mathbf{x},\mathbf{y},\omega_k) = F_{32}(\mathbf{x},\mathbf{y},\omega_k) = 0,$$

$$F_{11}(\mathbf{x},\mathbf{y},k) = \frac{\mu}{4\pi}\left[\frac{2}{r^2}\chi - \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{(y_1-x_1)^2}{r^2}\left(-\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{3}{r}\frac{\partial\chi}{\partial r} - \frac{6}{r^2}\chi\right)\right],$$

$$F_{22}(\mathbf{x},\mathbf{y},k) = \frac{\mu}{4\pi}\left[\frac{2}{r^2}\chi - \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{(y_2-x_2)^2}{r^2}\left(-\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{3}{r}\frac{\partial\chi}{\partial r} - \frac{6}{r^2}\chi\right)\right],$$

$$F_{12}(\mathbf{x},\mathbf{y},k) = \frac{\mu}{4\pi}\frac{(y_1-x_1)(y_2-x_2)}{r^2}\left(-\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{3}{r}\frac{\partial\chi}{\partial r} - \frac{6}{r^2}\chi\right), \tag{2}$$

$$F_{33}(\mathbf{x},\mathbf{y},k) = \frac{1}{4\pi\mu}\left[\lambda^2\frac{\partial^2\chi}{\partial r^2} + \frac{4\lambda(\lambda+\mu)}{r}\frac{\partial\chi}{\partial r} + \frac{2(\lambda^2+4\lambda\mu+4\mu^2)}{r^2}\chi - \right.$$

$$\left. -\lambda^2\frac{\partial^2\psi}{\partial r^2} - \frac{2(\lambda^2+2\lambda\mu+2\mu^2)}{r^2}\frac{\partial\psi}{\partial r}\right],$$

where $r^2 = (x_i - y_i)(x_i - y_i)$, λ and μ are Lamé constants and $l_1 = kr/c_1$, $l_2 = kr/c_2$. Further, $c_1 = \sqrt{(\lambda+2\mu)/\rho}$ and $c_2 = \sqrt{\mu/\rho}$ are the velocities of the longitudinal and transverse waves, respectively. Functions ψ and χ are given, for the static problem, by

$$\psi = \frac{(3-4\nu)}{4(1-\nu)}\frac{1}{r}, \quad \chi = \frac{1}{4(1-\nu)}\frac{1}{r}\ , \tag{3}$$

and, for the dynamic problem, by

$$\psi = \left(\frac{1}{l_2^2}+\frac{1}{l_2}+1\right)\frac{e^{-l_2}}{r} - \frac{c_2^2}{c_1^2}\left(\frac{1}{l_1^2}+\frac{1}{l_1}\right)\frac{e^{-l_1}}{r}, \quad \chi = \left(\frac{3}{l_2^2}+\frac{3}{l_2}+1\right)\frac{e^{-l_2}}{r} - \frac{c_2^2}{c_1^2}\left(\frac{3}{l_1^2}+\frac{3}{l_1}+1\right)\frac{e^{-l_1}}{r}\ .$$

$$\tag{4}$$

3. Regularization in the static case

Substituting the functions ψ and χ given by (3) into (2) we obtain the kernels in the form

$$F_{11} = \frac{1}{r^3} \frac{\mu}{4\pi(\lambda+2\mu)} \left[3\lambda \frac{(y_1-x_1)^2}{r^2} + 2\mu \right], \; F_{12} = \frac{1}{r^3} \frac{3\lambda\mu}{4\pi(\lambda+2\mu)} \frac{(y_1-x_1)(y_2-x_2)}{r^2} \; ,$$

$$F_{22} = \frac{1}{r^3} \frac{\mu}{4\pi(\lambda+2\mu)} \left[3\lambda \frac{(y_2-x_2)^2}{r^2} + 2\mu \right], \; F_{33} = \frac{1}{r^3} \frac{\mu(\lambda+\mu)}{2\pi(\lambda+2\mu)} \; .$$

Notice that all integrals with singularities in the integral equations (1) can be presented in the form

$$J_k^{l,m} = \int_{S_n} \frac{(x_1-y_1)^l(x_2-y_2)^m}{r^k} dS \; , \tag{5}$$

where S_n is the area of the element. These integrals may be regularized using the formula (7) given in [9].

3.1 Integral with the kernel r^{-3}

Let us consider the functions $f(\mathbf{x})$ and $\varphi(\mathbf{x})$ in the form

$$f(\mathbf{x}) = \frac{1}{r^3}, \; \varphi(\mathbf{x}) = 1 \, .$$

Using the formula (7) in [9] we obtain

$$J_3^{0,0} = F.P. \int_{S_n} \frac{dS}{r^3} = \int_{\partial S_n} \partial_n \frac{1}{r} dl = -\int_{\partial S_n} \frac{r_n}{r^3} dl \; ,$$

where $r_n = (x_\alpha - y_\alpha)n_\alpha$ and $\alpha = 1,2$.

3.2 Integral with the kernel $\dfrac{(x_\alpha - y_\alpha)^2}{r^5}$, $\alpha = 1,2$

Let us consider the function $f(\mathbf{x})$ and $\varphi(\mathbf{x})$ in the form

$$f(\mathbf{x}) = \frac{1}{r^5}, \; \varphi(\mathbf{x}) = (x_\alpha - y_\alpha)^2 \, .$$

Using formula (7) in [9] we obtain

$$J_5^{2,0} = F.P. \int_{S_n} \frac{(x_\alpha - y_\alpha)^2}{r^5} dS = \frac{2}{3} \int_{\partial S_n} \partial_n \frac{1}{r} dl - \frac{1}{3} \int_{\partial S_n} \partial_n \frac{(x_\alpha - y_\alpha)^2}{r^3} dl =$$

$$= \int_{\partial S_n} [\frac{(x_\alpha - y_\alpha)^2 r_n}{r^5} - \frac{2(x_\alpha - y_\alpha)n_\alpha}{3r^3} + \frac{2r_n}{3r^3}]dl \ .$$

3.3 Integral with the kernel $\dfrac{(x_1 - y_1)(x_2 - y_2)}{r^5}$

Let us consider the function $f(\mathbf{x})$ and $\varphi(\mathbf{x})$ in the form

$$f(\mathbf{x}) = \frac{1}{r^5} \ , \ \ \varphi(\mathbf{x}) = (x_1 - y_1)(x_2 - y_2) \ .$$

Using formula (7) in [9] we obtain

$$J_5^{1,1} = F.P.\int_{S_n} \frac{(x_1 - y_1)(x_2 - y_2)}{r^5} dS = -\frac{1}{3}\int_{\partial S_n} \partial_n \frac{(x_1 - y_1)(x_2 - y_2)}{r^3} dl =$$

$$= \frac{1}{3}\int_{\partial S_n} [\frac{3(x_1 - y_1)(x_2 - y_2)r_n}{r^5} - \frac{r_*}{r^3}]dl \ ,$$

where $r_* = x_1 n_2 + x_2 n_1$. As a result of the regularization we have the regular integrals over contour ∂S_n , instead of the hypersingular integrals over surface S_n , which can be easily calculated analytically or numerically.

4. Regularization in the dynamic case

Substituting the functions ψ and χ defined by (4) into (2), presenting the exponential functions by series and isolating singularities we obtain the following results for the integral kernels $F_{ij}(\mathbf{x},\mathbf{y},k)$

$$F_{11} = \frac{\mu^2}{2\pi(\lambda + 2\mu)}\frac{1}{r^3} + \frac{3\lambda\mu}{4\pi(\lambda + 2\mu)}\frac{(y_1 - x_1)^2}{r^5} +$$

$$+ \frac{\mu\omega_k^2}{8\pi}\left(\frac{1}{c_2^2} + \frac{c_2^2}{c_1^4}\right)\frac{1}{r} - \frac{\mu\omega_k^2}{8\pi}\frac{c_2^2}{c_1^4}\frac{(y_1 - x_1)^2}{r^3} - \frac{\mu}{2\pi}\frac{1}{r^3}\sum_{n=3}^{\infty}(-i\omega_k r)^n \frac{(n-1)}{n!(n+2)}\left[\frac{n}{c_2^n} + 2\frac{c_2^2}{c_1^{2+n}}\right] -$$

$$- \frac{\mu}{4\pi}\frac{(y_1 - x_1)^2}{r^5}\sum_{n=4}^{\infty}(-i\omega_k r)^n \frac{(n-1)(n-3)}{n!(n+2)}\left[\frac{(n-2)}{c_2^n} + 4\frac{c_2^2}{c_1^{2+n}}\right],$$

$$F_{12} = F_{21} = \frac{3\lambda\mu}{4\pi(\lambda + 2\mu)}\frac{(y_1 - x_1)(y_2 - x_2)}{r^5} - \frac{\mu\omega_k^2}{8\pi}\frac{c_2^2}{c_1^4}\frac{(y_1 - x_1)(y_2 - x_2)}{r^3} -$$

$$- \frac{\mu}{4\pi}\frac{(y_1 - x_1)(y_2 - x_2)}{r^5}\sum_{n=4}^{\infty}(-i\omega_k r)^n \frac{(n-1)(n-3)}{n!(n+2)}\left[\frac{(n-2)}{c_2^n} + 4\frac{c_2^2}{c_1^{2+n}}\right],$$

$$F_{22} = \frac{\mu^2}{2\pi(\lambda+2\mu)}\frac{1}{r^3} + \frac{3\lambda\mu}{4\pi(\lambda+2\mu)}\frac{(y_2-x_2)^2}{r^5} +$$

$$+ \frac{\mu\omega_k^2}{8\pi}\left(\frac{1}{c_2^2}+\frac{c_2^2}{c_1^4}\right)\frac{1}{r} - \frac{\mu\omega_k^2}{8\pi}\frac{c_2^2}{c_1^4}\frac{(y_2-x_2)^2}{r^3} - \frac{\mu}{2\pi}\frac{1}{r^3}\sum_{n=3}^{\infty}(-i\omega_k r)^n\,\frac{(n-1)}{n!(n+2)}\left[\frac{n}{c_2^n}+2\frac{c_2^2}{c_1^{2+n}}\right] -$$

$$- \frac{\mu}{4\pi}\frac{(y_2-x_2)^2}{r^5}\sum_{n=4}^{\infty}(-i\omega_k r)^n\,\frac{(n-1)(n-3)}{n!(n+2)}\left[\frac{(n-2)}{c_2^n}+4\frac{c_2^2}{c_1^{2+n}}\right],$$

$$F_{33} = \frac{\mu(\lambda+\mu)}{2\pi(\lambda+2\mu)}\frac{1}{r^3} + \frac{\omega_k^2}{8\pi\mu}\left(\frac{\mu^2}{c_2^2}+\left(2\lambda^2+4\lambda\mu+3\mu^2\right)\frac{c_2^2}{c_1^4}\right)\frac{1}{r} -$$

$$- \frac{1}{r^3 4\pi\mu}\sum_{n=3}^{\infty}(-i\omega_k r)^n\,\frac{(n-1)}{n!(n+2)}\left(\frac{4\mu^2(n-1)}{c_2^n}+\left[\lambda^2 n(n+2)+4\lambda\mu(n+2)+12\mu^2\right]\frac{c_2^2}{c_1^{2+n}}\right).$$

Notice that these equations contain divergent integrals of the type defined by (5). In addition to the hypersingular integrals that appear in the static case there exist some weakly singular integrals, which will be considered in the following.

4.1 Integral with the kernel r^{-1}

Let us consider the function $f(\mathbf{x})$ and $\varphi(\mathbf{x})$ in the form

$$f(\mathbf{x}) = \frac{1}{r}\ , \ \varphi(\mathbf{x}) = 1.$$

Using formula (7) in [9] we obtain

$$J_1^{0,0} = W.S.\int_{S_n}\frac{dS}{r} = \int_{\partial S_n}\partial_n r\, dl = \frac{1}{2}\int_{\partial S_n}\frac{r_n}{r}\,dl\ .$$

4.2 Integral with the kernel $\dfrac{(x_\alpha-y_\alpha)^2}{r^3}$ **,** $\alpha=1,2$

Let us consider the function $f(\mathbf{x})$ and $\varphi(\mathbf{x})$ in the form

$$f(\mathbf{x}) = \frac{(x_\alpha-y_\alpha)^2}{r^3}\ , \ \varphi(\mathbf{x}) = (x_\alpha-y_\alpha)^2\ .$$

Using formula (7) in [9] we obtain

$$J_3^{2,0} = W.S.\int_{S_n}\frac{(x_\alpha-y_\alpha)^2}{r^3}\,dS = \frac{2}{3}\int_{\partial S_n}\partial_n r\, dl - \frac{1}{3}\int_{\partial S_n}\partial_n\frac{(x_\alpha-y_\alpha)^2}{r}\,dl =$$

$$= \frac{1}{3}\int_{\partial S_n}\left[\frac{(x_\alpha-y_\alpha)^2 r_n}{r^3} - \frac{2(x_\alpha-y_\alpha)n_\alpha}{r} + \frac{2r_n}{r}\right]dl$$

4.3 Integral with the kernel $\dfrac{(x_1 - y_1)(x_2 - y_2)}{r^3}$

Let us consider the function $f(\mathbf{x})$ and $\varphi(\mathbf{x})$ in the form

$$f(\mathbf{x}) = \frac{1}{r^3} \ , \ \varphi(\mathbf{x}) = (x_1 - y_1)(x_2 - y_2) .$$

Using formula (7) in [9] we obtain

$$J_3^{1,1} = W.S. \int_{S_n} \frac{(x_1 - y_1)(x_2 - y_2)}{r^3} dS = -\frac{1}{3} \int_{\partial S_n} \partial_n \frac{(x_1 - y_1)(x_2 - y_3)}{r} dl =$$

$$= \frac{1}{3} \int_{\partial S_n} [\frac{3(x_1 - y_1)(x_2 - y_2)r_n}{r^3} - \frac{r_*}{r}] dl .$$

Notice that, after regularization, instead of the weakly singular integrals over surface S_n we have the regular integrals over contour ∂S_n, which can be easily calculated analytically or numerically.

5. Calculation of the divergent integrals over convex polygon

The divergent integrals of the type (5) have been transformed into regular integrals and may be easily calculated. For example, the integral $J_k^{0,0}$ for a circular area with the point $\mathbf{y}$ located in the center of circle leads to the following result

$$J_k^{0,0} = \frac{1}{(k-2)^2} \int_{\partial S_n} \partial_n \frac{1}{r^{k-2}} dl = \frac{1}{(k-2)^2} \int_0^{2\pi} \frac{d}{dr}\left(\frac{1}{r^{k-2}}\right) r d\varphi = -\frac{2\pi}{(k-2)r^{k-2}} ,$$

where polar coordinates with r is used. In the BEM application of the divergent integrals, it is necessary to calculate the above integrals over the triangular, rectangular or any polygonal elements. For that purpose these integrals must be transformed into a form more convenient for calculation.

Let us consider the case when the contour ∂S_n is a polygon with Q sides. All the calculations will be done using the local rectangular coordinate system with its origin at $\mathbf{y}$; the x_1 and x_2 axes are located in the plane of the polygon while the x_3 axis is perpendicular to this plane. Consider the q-th side of the polygon that starts from the vertex $(x_1(q), x_2(q))$. Then, the coordinates of an arbitrary point on this side are represented in the form

$$x_1(t) = x_1(q) - tn_2 \text{ and } x_2(t) = x_1(q) + tn_1 ,$$

where $\mathbf{n}(n_1, n_2)$ is a unit normal vector to the contour, $t \in [-\Delta_q, \Delta_q]$ is a parameter of integration and $2\Delta_q$ is the length of the q-th side. In addition, introduce the notations

$$r^2(t) = t^2 + 2tr_\tau(q) + r_\tau^2(q), \quad r_\tau(q) = -x_1(q)n_2(q) + x_2(q)n_1(q),$$

$$r^2(q) = x_1^2(q) + x_2^2(q), \quad r_n(q) = x_\alpha(q)n_\alpha(q), \quad r_*(q) = x_1(q)n_2(q) + x_2(q)n_1(q)$$

$$I_{m,l} = \int_{-\Delta_q}^{\Delta_q} \frac{t^l}{r^m(t)} dt = \int_{-\Delta_q}^{\Delta_q} \frac{t^l}{(t^2 + 2tr_\tau(q) + r_\tau^2(q))^{m/2}} dt .$$

Using these notations the integrals under consideration may be represented in a convenient form for the calculation as listed in the following.

5.1 Integrals with kernels of the type r^{-k}

$$J_1^{0,0} = \sum_{q=1}^{Q} r_n(q)I_{1,0}, \quad J_3^{0,0} = -\sum_{q=1}^{Q} r_n(q)I_{3,0} .$$

5.2 Integrals with kernels of the type $\dfrac{(x_\alpha - y_\alpha)^2}{r^k}$

$$J_3^{2,0} = \frac{1}{3}\sum_{q=1}^{Q} (r_n(q)(n_\beta^2(q)I_{3,2} - 2n_\beta(q)x_\alpha(q)I_{3,1} + x_\alpha^2(q)I_{3,0}) -$$

$$-2n_\alpha(q)(n_\beta(q)I_{1,1} - x_\alpha(q)I_{1,0}) + 2r_n(q)I_{1,0}),$$

$$J_5^{2,0} = \frac{1}{3}\sum_{q=1}^{Q} (r_n(q)(n_\beta^2(q)I_{5,2} - 2n_\beta(q)x_\alpha(q)I_{5,1} + x_\alpha^2(q)I_{5,0}) -$$

$$-2n_\alpha(q)(n_\beta(q)I_{3,1} - x_\alpha(q)I_{3,0}) - 2r_n(q)I_{3,0}) .$$

5.3 Integrals with kernels of the type $\dfrac{(x_1 - y_1)(x_2 - y_2)}{r^k}$

$$J_3^{1,1} = \frac{1}{3}\sum_{q=1}^{Q} (r_n(q)(-n_1(q)n_2(q)I_{3,2} + r^-(q)I_{3,1} + x_1(q)x_2(q)I_{3,0}) -$$

$$J_5^{1,1} = \frac{1}{3}\sum_{q=1}^{Q} (3r_n(q)(-n_1(q)n_2(q)I_{5,2} + r^-(q)I_{5,1} + x_1(q)x_2(q)I_{5,0}) -$$

$$- (n_1^2(q) - n_2^2(q))I_{3,1} - r_*(q)I_{3,0})) .$$

Conclusions

Equations presented in [9] have been applied for the regularization of the weakly singular and hypersingular integrals, which arise when the BIE methods are used to solve problems in fracture mechanics. The divergent integrals over arbitrary convex polygon have been transformed to regular contour integrals that can be easily calculated analytically or numerically.

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Appendix

The formula (7) in [9] is reproduced here for the convenience of the readers.

$$F.P. \int_V \frac{\varphi(\mathbf{x})}{r^m} dV = \sum_{i=0}^{k-1} (-1)^{i+1} \int_{\partial V} [\Delta^{k-i-1}\varphi(\mathbf{x})\partial_n \frac{P_i}{r^{m-2i}} - \frac{P_i}{r^{m-2i}} \partial_n \Delta^{k-i-1}\varphi(\mathbf{x})] dS +$$

$$+ (-1)^k \int_V \frac{1}{r^{m-2k}} \Delta^{k+1}\varphi(\mathbf{x})] dV \ .$$