Regularization of the Divergent Integrals
I. General Consideration

V.V. Zozulya
Centro de Investigacion Cientifica de Yucatan A.C.,
Calle 43, No 130, Colonia Chuburná de Hidalgo, C.P. 97200,
Mérida, Yucatán, México. E-mail: *zozulya@cicy.mx*

Abstract

This article considers weakly singular, singular and hypersingular integrals, which arise when the boundary integral equation (BIE) methods are used to solve problems in science and engineering. For their regularization, an approach based on the theory of distribution and application of the Green theorem has been used. The expressions, which allow an easy calculation of the weakly singular, singular and hypersingular integrals, have been constructed. Such approach may be easily generalized and applied to the calculation of multidimensional integrals with singularities of various types.

Key words: weakly singular, singular, hypersingular integrals, regularization, boundary integral equations

1. Introduction

In [1] it was shown that divergent integrals and integral operators with divergent kernels are widely used in mathematics, applied science and engineering. Their correct mathematical interpretation has been done in terms of the theory of distributions (generalized functions) by Gel'fand and Shilov in [3]. Also interesting interpretation of the divergent integrals has been given by Courant in [2]. In mathematics weakly singular (*WS*) and singular integrals and integral operators with such kernels have a well-established theoretical basis [7]. For example, the weakly singular integrals are considered as improper integrals, the singular integrals are considered in the sense of Cauchy principal value (*PV*). The hypersingular integrals had been considered by Hadamard as finite part integrals (*FP*) in [6]. The theory of distributions provides a unified approach for the study of the divergent integrals and integral operators with kernels containing different kind of singularities. Mathematical methodology of this approach is well known and widely discussed in the mathematical literature, but until recently, it had not been used for the numerical solution of the BIE with divergent integrals. One-dimensional (1-D) and multi-dimensional divergent integrals can also be calculated using the same method. For example, two-dimensional (2-D) hypersingular integrals that arise from the BIE solution of the 3-D static and dynamic problems of fracture mechanics have been considered in [9,10].

In the present paper, the approach for the divergent integral regularization based on the theory of distribution and Green theorem is developed. Equations that enable easy calculation of the weakly singular, singular and hypersingular integrals over any convex polygonal area are presented here.

2. Boundary integral equations

Let V be an open bounded subset of the three-dimensional Euclidean space R^3 with a $C^{0,1}$ Lipschitzian regular boundary ∂V. The region V is occupied by an elastic body in its undeformed state. The physical processes in the region V are descried by the vector function u_i that in general depends on the spatial coordinate $\mathbf{x}$ and time t. Time dependent problems may be considered in the time domain, Laplace transformed domain and the frequency domain.

In [5, 8] it was shown that the BIE that appears in many engineering applications may be written, on a smooth boundary, in the following form

$$\begin{aligned} \mp \tfrac{1}{2} u_i(\mathbf{y},\bullet) &= \int_V (p_j(\mathbf{x},\bullet) * U_{ij}(\mathbf{x}-\mathbf{y},\bullet) + u_j(\mathbf{x},\bullet) * W_{ij}(\mathbf{x},\mathbf{y},\bullet))dS + u_i^*(\mathbf{y},\bullet) \\ \mp \tfrac{1}{2} p_i(\mathbf{y},\bullet) &= \int_V (p_j(\mathbf{x},\bullet) * K_{ij}(\mathbf{x},\mathbf{y},\bullet) + u_j(\mathbf{x},\bullet) * F_{ij}(\mathbf{x},\mathbf{y},\bullet))dS + p_i^*(\mathbf{y},\bullet) \end{aligned} , \tag{1}$$

with $u_i^*(\mathbf{y},\bullet) = \int_V b_j(\mathbf{x},\bullet) * U_{ij}(\mathbf{x}-\mathbf{y},\bullet)dV$, $p_i^*(\mathbf{y},\bullet) = \int_V b_j(\mathbf{x},\bullet) * K_{ij}(\mathbf{x},\mathbf{y},\bullet)dV$.

Here $\bullet$ indicates t for the time domain, k for the Laplace transformed domain, ω for the frequency domain formulations and zero for the statically problems, respectively. The plus and minus signs in (1) are used for the interior and exterior problems, respectively. Sign $*$ indicates the convolution,

$$f(t) * g(t) = \int_{\Im} f(\tau) g(t-\tau) d\tau ,$$

in the time domain BIE formulation and a multiplication of functions otherwise. The summation convention is used in (1) for repeated indices. In the case of scalar problem indices are omitted.

The kernels $U_{ij}(\mathbf{x}-\mathbf{y},\bullet)$, $W_{ij}(\mathbf{x},\mathbf{y},\bullet)$, $K_{ij}(\mathbf{x},\mathbf{y},\bullet)$ and $F_{ij}(\mathbf{x},\mathbf{y},\bullet)$ in the BIE (1) are fundamental solutions for the differential equations that correspond to the problem under consideration. The expressions of these kernels are well known and can be found in most books on the BEM (see also [4, 5]). As it is well known the fundamental solutions for systems of differential equations of the second order have the same singularity when $\mathbf{y} \to \mathbf{x}$ (see for references [2, 4, 5]). In the 2-D case we have

$$U_{ji}(\mathbf{x}-\mathbf{y}) \to ln(r), W_{ji}(\mathbf{x},\mathbf{y}) \to r^{-1}, K_{ji}(\mathbf{x},\mathbf{y}) \to r^{-1}, F_{ji}(\mathbf{x},\mathbf{y}) \to r^{-2} ,$$

where $r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ is a distance between the points **x** and **y**.
In the 3-D case we have

$$U_{ji}(\mathbf{x}-\mathbf{y}) \to r^{-1}, W_{ji}(\mathbf{x},\mathbf{y}) \to r^{-2}, K_{ji}(\mathbf{x},\mathbf{y}) \to r^{-2}, F_{ji}(\mathbf{x},\mathbf{y}) \to r^{-3},$$

where $r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$.
The kernels $U_{ji}(\mathbf{x}-\mathbf{y})$ are weakly singular (*WS)*, the kernels $W_{ji}(\mathbf{x},\mathbf{y})$ and $K_{ji}(\mathbf{x},\mathbf{y})$ are singular and the kernels $F_{ji}(\mathbf{x},\mathbf{y})$ are hypersingular. The integrals with such kernels are divergent and they can not be calculated using Gauss formulas for example. These integrals need a special consideration in order to have some mathematical sense.

3. Divergent integrals and distributions

We will consider the concept of the definite integral based of the theory of distribution. To clarify consideration we will study 1-D divergent integrals first. Then we will extend definitions and methods for 2-D divergent integrals.

3.1 1-D divergent integrals

Let a hypersingular function of one variable $f(x)$ be defined in the region $x \in V = [-a, a]$. All its singularities are concentrated in the smaller region $V^{\varepsilon} = [-\varepsilon, \varepsilon] \subset V$ so that, in the region $V^0 = V / V^{\varepsilon} = [-a, -\varepsilon] \cup [\varepsilon, a]$, the function $f(x)$ is regular. Let us consider a definite integral

$$I_0 = \int_{-\varepsilon}^{\varepsilon} f(x)dx$$

and inquire the meaning of I_0 for this hypersingular function, which cannot be answered by the classical approach. In order to consider this integral in the sense of the distribution, we introduce a function $g(x)$, such that

$$f(x) = \frac{d^k g(x)}{dx^k} .$$

This equation has to be considered in the classical sense in the region V^0 and in the sense of a distribution in the region V^{ε}. For details one can refer to [2]. We introduce the test function $\varphi(x) \in C^{\infty}(R)$, such that $\varphi(x) = 1$, $\forall x \in V^{\varepsilon}$ and $\varphi(x) = 0$, $\forall x \notin V$. Function $\varphi(x)$ is extended arbitrary to the region V^0. In this case, its derivatives are equal to zero at the end points of the regions V^{ε} and V, i.e.,

$$\frac{d^k\varphi(x)}{dx^k}=0\ ,\ x\in\partial V^{\varepsilon}\cup\partial V=\{-\varepsilon,\varepsilon\}\cup\{-a,a\}\,.$$

We consider the scalar product

$$(f,\varphi)=\int_{-a}^{a}f(x)\varphi(x)dx=\int_{-a}^{a}\varphi(x)\frac{d^k g(x)}{dx^k}dx\,.$$

Since the derivatives of the test function $\varphi(x)$ is equal zero on the boundary $\partial V^0=\{-\varepsilon,\varepsilon\}\cup\{-a,a\}$, the integration by parts gives

$$\int_{-a}^{a}\varphi(x)\frac{d^k g(x)}{dx^k}dx=(-1)^k\int_{-a}^{a}g(x)\frac{d^k\varphi(x)}{dx^k}dx=(-1)^k\left(\int_{-a}^{-\varepsilon}g(x)\frac{d^k\varphi(x)}{dx^k}dx+\int_{\varepsilon}^{a}g(x)\frac{d^k\varphi(x)}{dx^k}dx\right).$$

The integration by parts in reverse order for the last integrals leads to the result

$$\left(\int_{-a}^{-\varepsilon}g(x)\frac{d^k\varphi(x)}{dx^k}dx+\int_{\varepsilon}^{a}g(x)\frac{d^k\varphi(x)}{dx^k}dx\right)=$$
$$=(-1)^k\left(\int_{-a}^{-\varepsilon}\varphi(x)\frac{d^k g(x)}{dx^k}dx+\int_{\varepsilon}^{a}\varphi(x)\frac{d^k g(x)}{dx^k}dx\right)+\left.\frac{d^{k-1}g(x)}{dx^{k-1}}\right|_{x=-\varepsilon}^{x=\varepsilon}\,.$$

Taking into account that

$$\int_{V^{\varepsilon}}f(x)\varphi(x)dx=\int_{V}f(x)\varphi(x)dx-\int_{V^{o}}f(x)\varphi(x)dx$$

we will find the finite part of the divergent integral according to Hadamard in the form

$$I_0=F.P.\int_{-\varepsilon}^{\varepsilon}f(x)dx=\left.\frac{d^{k-1}g(x)}{dx^{k-1}}\right|_{x=-\varepsilon}^{x=\varepsilon}\,. \tag{2}$$

We can use this equation to calculate weakly singular, singular and hypersingular integrals. For the regular functions this is a usual formula from integral calculus which connects infinite and finite integrals. Obviously for $k=1$ we have

$$F.P.\int_{-a}^{a}f(x)dx=g(x)\Big|_{x=-a}^{x=a}\,, \tag{3}$$

which is the well known Leibniz's formula for the definite integral.

3.2 2-D divergent integrals

We will consider now the 2-D divergent integrals in the sense of distribution and find analogies for the equations (2) and (3). The symbol ∂_i^{k-1} in the multidimensional case may be represented in the form $\partial_i^{k-1} = \partial_i^{-1}\partial_i^k$, were symbol ∂_i^{-1} is defined as an inverse operator for the operator of partial derivative ∂_i and also as an indefinite integral operator with respect to x_i.

In the same way as in the 1-D case we consider a hypersingular function $f(\mathbf{x})$ of 2 variables $\mathbf{x} = (x_1, x_2)$ that is defined in the region $\mathbf{x} \in V$. All its singularities are concentrated in the region $V^\varepsilon \subset V$ and in the region $V^0 = V / V^\varepsilon$ the function is regular. We suppose that the boundaries of the regions V and V^0 satisfy some special conditions of smoothness, which are discussed in any standard courses of analysis. Let as consider a definite 2-D integral

$$I_0 = \int_{V^\varepsilon} f(\mathbf{x}) d\mathbf{x} .$$

Once again the classical approach can not provide the meaning of I_0 in the 2-D case. To consider this integral in the sense of the distribution, introduce the function $g(\mathbf{x})$, such that

$$f(\mathbf{x}) = \Delta^k g(\mathbf{x}) ,$$

where $\Delta^k = \partial_1^{2k} + \partial_2^{2k}$, which is called the k – dimensional Laplace's operator.

This representation of the function $f(\mathbf{x})$ has to be considered in the classical sense in the region V^0 and in the sense of distribution in the region V^ε. We also introduce test function $\varphi(x) \in C^\infty(R^2)$, such that $\varphi(\mathbf{x}) = 1$, $\forall \mathbf{x} \in V^\varepsilon$ and $\varphi(\mathbf{x}) = 0$, $\forall \mathbf{x} \notin V$. Function $\varphi(\mathbf{x})$ is extended arbitrary to the region V^0. In this case, its derivatives are equal to zero at the boundary of the regions V^ε and V, i.e.,

$$\Delta^k \varphi(\mathbf{x}) = 0 \ , \ \mathbf{x} \in \partial V^\varepsilon \cup \partial V .$$

We consider the scalar product

$$(f, \varphi) = \int_{V^\varepsilon} f(\mathbf{x})\varphi(\mathbf{x}) d\mathbf{x} = \int_{V^\varepsilon} \varphi(\mathbf{x}) \Delta^k g(\mathbf{x}) d\mathbf{x} .$$

Since the derivatives of the test function $\varphi(\mathbf{x})$ is equal zero on $\partial V^0 = \partial V^\varepsilon \cup \partial V$, the application of the Green theorem gives

$$\int_V [\varphi(\mathbf{x})\Delta^k g(\mathbf{x}) - (-1)^k g(\mathbf{x})\Delta^k \varphi(\mathbf{x})] dV =$$

$$= \sum_{i=0}^{k-1} (-1)^{i+1} \int_{\partial V} [\varphi(\mathbf{x})\partial_n \Delta^{k-i-1} g(\mathbf{x}) - g(\mathbf{x})\partial_n \Delta^{k-i-1} \varphi(\mathbf{x})] dS .$$

Here, $\partial_n = n_i \partial_i$ is the normal derivative on the surface with respect to $\mathbf{x}$ and $n_i(\mathbf{x})$ is a unit normal to the surface. The integration by parts in reverse order for the last integrals above leads to the result

$$\int_{V^0} g(\mathbf{x}) \Delta^k \varphi(\mathbf{x}) dV = \int_{V^0} \varphi(\mathbf{x}) \Delta^k g(\mathbf{x}) dV - \int_{\partial V^0} \partial_n \Delta^{k-1} g(\mathbf{x}) dS \ .$$

Taking into account that

$$\int_{V^\varepsilon} f(x)\varphi(x)dx = \int_{V} f(x)\varphi(x)dx - \int_{V^o} f(x)\varphi(x)dx$$

we will find the finite part of the divergent integral according to Hadamard in the form

$$F.P.\int_{V^\varepsilon} f(\mathbf{x}) dV = \int_{\partial V^\varepsilon} \partial_n \Delta^{k-1} g(\mathbf{x}) dS \tag{4}$$

or, for $k = 1$, we get

$$F.P.\int_{V^\varepsilon} f(\mathbf{x}) dV = \int_{\partial V^\varepsilon} \partial_n g(\mathbf{x}) dS \ . \tag{5}$$

4. Regularization of divergent integrals

We will apply here the approach demonstrated in the previous section to the regularization of wide class of divergent integrals which arise in engineering applications, especially in the BIE methods.

4.1 1-D representation

Let us consider the function $f(x)$ in the form

$$f(x) = \frac{1}{r^m} \ .$$

It can be represented as

$$f(x) = \frac{d^k g(x)}{dx^k}$$

where $g(x) = \frac{P_k}{r^{m-k}}$, and $P_k = (-1)^k \prod_{i=0}^{k-1} \frac{1}{(m+i)}$ for $k, m > 1$.

Let us consider the scalar product

$$(f,\varphi) = \int_{-a}^{a} f(x)\varphi(x)dx = \int_{-a}^{a} \varphi(x) \frac{d^k g(x)}{dx^k} dx \ .$$

The test function here is $\varphi(x) \in C^{\infty}(V)$. In contrast to Section 3, no boundary conditions are implied. Then using the integration by parts k times we obtain

$$\int_{-a}^{a}\left(\varphi(x)\frac{d^k}{dx^k}\frac{1}{r^{m-k}} - (-1)^k \frac{1}{r^{m-k}}\frac{d^k\varphi(x)}{dx^k}\right)dx = \sum_{i=0}^{k-1}(-1)^{i+1}\frac{d^i}{dx^i}\frac{1}{r^{m-k}}\frac{d^{k-1-i}\varphi(x)}{dx^{k-1-i}}\Bigg|_{x=-a}^{x=a} .$$

Operations of differentiation and integration here are considered in the sense of distribution. As a result we obtain a formula for the calculation of 1-D divergent integrals in the sense of Hadamard's finite part in the form

$$F.P.\int_{-a}^{a}\frac{\varphi(x)}{r^m}dx = \sum_{i=0}^{k-1}(-1)^{i+1}\frac{d^i}{dx^i}\frac{P_i}{r^{m-k}}\frac{d^{k-1-i}\varphi(x)}{dx^{k-1-i}}\Bigg|_{x=-a}^{x=a} + (-1)^k\int_{-a}^{a}\frac{P_k}{r^{m-k}}\frac{d^k\varphi(x)}{dx^k} . \quad (6)$$

4.2 2-D representation

Let us consider the function $f(\mathbf{x})$ in the form

$$f(\mathbf{x}) = \frac{1}{r^m} .$$

It can be represented as

$$f(\mathbf{x}) = \Delta^k g(\mathbf{x}) ,$$

where $g(\mathbf{x}) = \frac{P_k}{r^{m-2k}}$ and $P_k = (-1)^k \prod_{i=0}^{k-1}\frac{1}{(m+2i)^2}$ for $k, m > 1$.

Let us consider the scalar product

$$(f, \varphi) = \int_{V^{\varepsilon}} f(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} = \int_{V^{\varepsilon}} \varphi(\mathbf{x})\Delta^k g(\mathbf{x})d\mathbf{x} .$$

The test function here is $\varphi(\mathbf{x}) \in C^{\infty}(V)$. In contrast to Section 3, no boundary conditions are implied. Then using the Green theorem k times we obtain

$$\int_V [\varphi(\mathbf{x})\Delta^k \frac{1}{r^{m-2}} - (-1)^k \frac{1}{r^{m-2}}\Delta^k\varphi(\mathbf{x})]dV =$$
$$= \sum_{i=0}^{k-1}(-1)^{i+1}\int_{\partial V}[\varphi(\mathbf{x})\partial_n\Delta^{k-i-1}\frac{1}{r^{m-2}} - \frac{1}{r^{m-2}}\partial_n\Delta^{k-i-1}\varphi(\mathbf{x})]dS .$$

Operations of differentiation and integration here are considered in the sense of distribution and we obtain a formula for the calculation 2-D divergent integrals in the sense of Hadamard's finite part in the form

$$F.P.\int_V \frac{\varphi(\mathbf{x})}{r^m}dV = \sum_{i=0}^{k-1}(-1)^{i+1}\int_{\partial V}[\Delta^{k-i-1}\varphi(\mathbf{x})\partial_n \frac{P_i}{r^{m-2i}} - \frac{P_i}{r^{m-2i}}\partial_n\Delta^{k-i-1}\varphi(\mathbf{x})]dS +$$
$$+(-1)^k\int_V \frac{1}{r^{m-2k}}\Delta^{k+1}\varphi(\mathbf{x})]dV \ . \quad (7)$$

Notice that if the integral on the right hand side is still divergent, then k has to be increased.

Conclusions

General formulas for the regularization of the divergent integrals of the type

$$\int_V \frac{\varphi(\mathbf{x})}{r^m}dV$$

for 1-D and 2-D have been derived. The approach based on the theory of distribution has been used. These formulas allow us to calculate the weakly singular, singular and hypersingular integrals in a unified fashion. This method may be easily extended to calculate various multidimensional divergent integrals.

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