2D Fundamental Solutions for the General Anisotropic Solids with Computer Codes

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Abstract
Basic equations with detailed derivation for the two-dimensional static fundamental solutions are presented for the general anisotropic solids. The displacement, stress and traction solutions for the line force and dislocation fundamental solutions are given along with FORTRAN 90 computer codes.

1 Introduction

Two-dimensional fundamental solutions for the general anisotropic elastic solids, defined by 21 elastic constants, are derived. The generalized plane strain is assumed. After a short review of Stroh's complex variable formalism for 2D anisotropic elasticity (Stroh [1, 2], Ting [3], Suo [4] and Ni and Nemat-Nasser [5]) the fundamental solutions for the line force and line dislocation are derived following Denda [6] and Denda and Marante [7]. The explicit form of the fundamental solutions are given for the displacement, traction and stress components for the line force and dislocation. Brief description of the downloadable FORTRAN 90 codes is given.

2 Basic Equations

For the generalized plane strain anisotropic elasticity problems considered the displacement $\mathbf{u} = \{u_1, u_2, u_3\}^T$ depends only on two coordinates x_1 and x_2. This is the case if the geometry and the loading do not vary in the x_3-axis (i.e., out-of-plane) direction. If the anisotropic material has a symmetry plane parallel to the x_1x_2-plane, then the in-plane (u_1 and u_2) and the out-of-plane (u_3) deformations become uncoupled. Otherwise, full coupling exists between two deformations. The latter is assumed in this paper including the former as a special case. The equilibrium equation, in the absence of the body force, is given in a vector form by

$$\frac{\partial \boldsymbol{\sigma}_1}{\partial x_1} + \frac{\partial \boldsymbol{\sigma}_2}{\partial x_2} = \mathbf{0}, \tag{1}$$

where $\boldsymbol{\sigma}_1 = \{\sigma_{11}, \sigma_{12}, \sigma_{13}\}^T$ and $\boldsymbol{\sigma}_2 = \{\sigma_{21}, \sigma_{22}, \sigma_{23}\}^T$ are the stress vectors. In describing the field variables we replace a pair of suffices ij by a single suffix M following the convention $(11 \to 1)$, $(22 \to 2)$, $(33 \to 3)$, $(23 \to 4)$, $(31 \to 5,)$ $(12 \to 6)$. The non zero strain components are given by

$$e_1 = \frac{\partial u_1}{\partial x_1}, \qquad e_2 = \frac{\partial u_2}{\partial x_2}, \qquad e_4 = \frac{\partial u_3}{\partial x_2}, \qquad e_5 = \frac{\partial u_3}{\partial x_1}, \qquad e_6 = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}. \tag{2}$$

The strain-stress relations are given by

$$e_M = \sum_{N=1}^{6} S_{MN}\, \sigma_N \quad \text{(M,N = 1,2,4,5,6)}, \tag{3}$$

where S_{MN} is the reduced compliance defined by

$$S_{MN} = s_{MN} - (s_{M3}\, s_{3N})/s_{33} \quad \text{(M,N = 1,2,4,5,6)} \tag{4}$$

in terms of the compliance s_{ij} $(i, j = 1, 2, 3, 4, 5, 6)$. In this paper no summation convention is used for repeated indices unless mentioned otherwise. The compatibility equations are given by

$$\frac{\partial^2 e_2}{\partial x_1^2} + \frac{\partial^2 e_1}{\partial x_2^2} - \frac{\partial^2 e_6}{\partial x_1 \partial x_2} = 0, \qquad \frac{\partial e_4}{\partial x_1} - \frac{\partial e_5}{\partial x_2} = 0. \tag{5}$$

If we introduce a real-valued stress function vector $\boldsymbol{\phi} = \{\phi_1, \phi_2, \phi_3\}^T$ such that

$$\boldsymbol{\sigma}_1 = -\frac{\partial \boldsymbol{\phi}}{\partial x_2}, \qquad \boldsymbol{\sigma}_2 = \frac{\partial \boldsymbol{\phi}}{\partial x_1}, \tag{6}$$

then the equilibrium equation (1) is automatically satisfied.

Lekhnitskii [8] and Eshelby et al. [9] have shown that the solution of the generalized plane strain problem can be represented by three functions $f_1(z_1)$, $f_2(z_2)$, $f_3(z_3)$, each of which is analytic in its argument $z_\alpha = x_1 + p_\alpha x_2$. Here p_α are three distinct complex numbers: roots of the sixth-order polynomial characteristic equation (7). The arguments $z_\alpha = x_1 + p_\alpha x_2 (\alpha = 1, 2, 3)$ are called the generalized complex variables. Lekhnitskii [8] has assumed the stress function vector of the form

$$\boldsymbol{\phi} = \mathbf{l} f(x_1 + p x_2),$$

where $\mathbf{l} = \{L_1, L_2, L_3\}^T$, which is substituted in the compatibility equations (5). This results in the sixth-order characteristic equation in p

$$d^{(4)}(p)\, d^{(2)}(p) - d^{(3)}(p)\, d^{(3)}(p) = 0, \tag{7}$$

where

$$\begin{aligned} d^{(4)}(p) &= p^4 S_{11} - 2p^3 S_{16} + p^2(2S_{12} + S_{66}) - 2pS_{26} + S_{22}, \\ d^{(3)}(p) &= p^3 S_{15} - p^2(S_{14} + S_{56}) + p(S_{25} + S_{46}) - S_{24}, \\ d^{(2)}(p) &= p^2 S_{55} - 2pS_{45} + S_{44}. \end{aligned}$$

Lekhnitskii [8] has shown that (7) has no real roots and has three pairs of conjugate complex roots $p_1,\ \overline{p}_1,\ p_2,\ \overline{p}_2,\ p_3,\ \overline{p}_3$. The imaginary part of p_α $(\alpha = 1, 2, 3)$ is assumed, without loss of generality, positive. It is also assumed that the three roots p_1 , p_2, p_3 are distinct. In the numerical treatment the coincident roots can be made distinct by slightly perturbing compliance coefficients. The matrix

$$\mathbf{L} = [\mathbf{l_1}, \mathbf{l_2}, \mathbf{l_3},] = \begin{bmatrix} -p_1 L_{21} & -p_2 L_{22} & -p_3 l_3 L_{33} \\ L_{21} & L_{22} & l_3 L_{33} \\ l_1 L_{21} & l_2 L_{22} & L_{33} \end{bmatrix}, \tag{8}$$

where

$$l_\alpha = \frac{d^{(3)}(p_\alpha)}{d^{(2)}(p_\alpha)} \quad (\alpha = 1, 2) \qquad l_3 = \frac{d^{(3)}(p_3)}{d^{(4)}(p_3)}, \tag{9}$$

is obtained from the compatibility equations. The subsequent integration of the strain components, to get the displacement components, introduces the matrix

$$\mathbf{A} = [\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}], \tag{10}$$

with

$$\mathbf{a}_\alpha = \left\{ \begin{array}{c} A_{1\alpha} \\ A_{2\alpha} \\ A_{3\alpha} \end{array} \right\} = \left[\begin{array}{ccc} s_{16} - s_{11}p_\alpha, & s_{12}, & s_{14} - s_{15}p_\alpha \\ \frac{s_{26} - s_{21}p_\alpha}{p_\alpha}, & \frac{s_{22}}{p_\alpha}, & \frac{s_{24} - s_{25}p_\alpha}{p_\alpha} \\ s_{56} - s_{51}p_\alpha, & s_{52}, & s_{54} - s_{55}p_\alpha \end{array} \right] \left\{ \begin{array}{c} L_{1\alpha} \\ L_{2\alpha} \\ L_{3\alpha} \end{array} \right\}. \tag{11}$$

Eshelby et al. [9] have assumed the displacement vector of the form $\mathbf{u} = \mathbf{a}f(x_1 + px_2)$, where $\mathbf{a} = \{A_1, A_2, A_3\}^T$, which is substituted in the equilibrium equation (1) written in terms of the displacement. This leads to another sixth-order polynomial characteristic equation. These two approaches are equivalent. For the BEM formulation Denda [6] and Denda and Marante [7] have adopted Lekhnitskii's approach which gives the explicit expression for the elements of the $\mathbf{A}$ matrix in terms of those of $\mathbf{L}$ matrix. Notice that $\mathbf{L}$ and $\mathbf{A}$ given above include the case when the in-plane and out-of-plane deformations are decoupled. In this case ($s_{14} = s_{15} = s_{24} = s_{25} = s_{46} = s_{56} = 0$) we have

$$\mathbf{L} = \left[\begin{array}{ccc} -p_1 L_{21} & -p_2 L_{22} & 0 \\ L_{21} & L_{22} & 0 \\ 0 & 0 & L_{33} \end{array} \right], \tag{12}$$

and

$$\mathbf{A} = \left[\begin{array}{ccc} (s_{11}{p_1}^2 - p_1 s_{16} + s_{12})L_{21} & (s_{11}{p_2}^2 - p_2 s_{16} + s_{12})L_{22} & 0 \\ (s_{21}p_1 - s_{26} + \frac{s_{22}}{p_1})L_{21} & (s_{21}p_2 - s_{26} + \frac{s_{22}}{p_2})L_{22} & 0 \\ 0 & 0 & (s_{54} - s_{55}p_3)L_{33} \end{array} \right]. \tag{13}$$

Note that for each characteristic root p_α we can determine vectors $\mathbf{l}_\alpha$ and $\mathbf{a}_\alpha$ up to an arbitrary multiplying factor. The solution may be normalized by setting

$$L_{21} = L_{22} = L_{33} = 1 \tag{14}$$

in (8) and (12). The alternative normalization, adopted in this paper, is based on the relations [1, 2]

$$\begin{aligned} \mathbf{L}^T\mathbf{A} + \mathbf{A}^T\mathbf{L} &= \left[\begin{array}{ccc} 2\sum_{i=1}^{3} L_{i1}A_{i1} & 0 & 0 \\ 0 & 2\sum_{i=1}^{3} L_{i2}A_{i2} & 0 \\ 0 & 0 & 2\sum_{i=1}^{3} L_{i3}A_{i3} \end{array} \right], \\ \mathbf{L}^T\bar{\mathbf{A}} + \mathbf{A}^T\bar{\mathbf{L}} &= \mathbf{0}, \end{aligned} \tag{15}$$

where $\mathbf{0}$ is the 3×3 zero matrix and a bar and the superscript T indicate the complex conjugate and the transpose, respectively. The normalization is introduced by setting

$$2\sum_{i=1}^{3} L_{i\alpha}A_{i\alpha} = 1 \qquad (\alpha = 1, 2, 3). \tag{16}$$

This gives the orthogonality relations

$$\begin{aligned} \mathbf{L}^T\mathbf{A} + \mathbf{A}^T\mathbf{L} &= \mathbf{I} = \bar{\mathbf{L}}^T\bar{\mathbf{A}} + \bar{\mathbf{A}}^T\bar{\mathbf{L}} \\ \mathbf{L}^T\bar{\mathbf{A}} + \mathbf{A}^T\bar{\mathbf{L}} &= \mathbf{0} = \bar{\mathbf{L}}^T\mathbf{A} + \bar{\mathbf{A}}^T\mathbf{L}, \end{aligned} \tag{17}$$

where $\mathbf{I}$ is the 3×3 identity matrix. A through investigation on these matrices, $A_{i\alpha}$ and $L_{i\alpha}$, has been given by Stroh [1, 2], Ting [3], Suo [4] and Ni and Nemat-Nasser [5].

Using $A_{i\alpha}$ and $L_{i\alpha}$ we can represent the displacement u_i, stress σ_{ij} and the stress function ϕ_i in the form

$$\begin{aligned} u_i(z) &= 2\Re\left[\sum_{\alpha=1}^{3} A_{i\alpha}f_\alpha(z_\alpha)\right], \qquad \phi_i(z) = 2\Re\left[\sum_{\alpha}^{3} L_{i\alpha}f_\alpha(z_\alpha)\right] \\ \sigma_{2i}(z) &= 2\Re\left[\sum_{\alpha=1}^{3} L_{i\alpha}f'_\alpha(z_\alpha)\right], \qquad \sigma_{1i}(z) = -2\Re\left[\sum_{\alpha=1}^{3} L_{i\alpha}p_\alpha f'_\alpha(z_\alpha)\right], \end{aligned} \tag{18}$$

where $(\ )'$ indicates the derivative with respect to the argument of the function and $\Re$ is the real part of a complex variable.

3 Fundamental Solutions

3.1 Derivation

Consider a point force and a dislocation of magnitudes $\mathbf{r} = \{r_1, r_2, r_3\}^T$ and $\mathbf{b} = \{b_1, b_2, b_3\}^T$, respectively located at $\xi = \eta_1 + i\eta_2$ in the z-plane and introduce an arbitrary circuit Γ around ξ, which induces an additional circuit Γ_α in each of the z_α-plane as shown in Figure 1. We determine the complex potential function vector $\{f_1(z_1), f_2(z_2), f_3(z_3)\}^T$, such that the force resultant and the displacement jump around the circuit Γ are $-\mathbf{r}$ and $\mathbf{b}$, respectively. First notice that the resultant force is obtained by integrating the traction t_k around the circuit to get

$$\begin{aligned} r_k &= \int_\Gamma t_k ds = \int_\Gamma \sigma_{jk}n_j ds = \int_\Gamma (\sigma_{1k}n_1 + \sigma_{2k}n_2)ds \\ &= \int_\Gamma \left\{\left(\frac{\partial\phi_k}{\partial x_2}\right)\frac{dx_2}{ds} + \left(\frac{\partial\phi_k}{\partial x_1}\right)\frac{dx_1}{ds}\right\} ds \\ &= -\left[\phi_k\right]_I^F, \end{aligned} \tag{19}$$

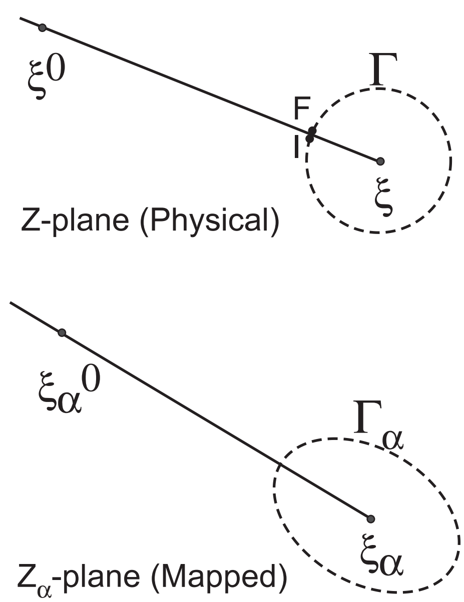

Figure 1: Physical (z) and mapped $(z_\alpha; \alpha = 1, 2, 3)$ planes. The source point ξ is mapped to ξ_α. Images $\xi^0_\alpha(\alpha = 1, 2, 3)$ of the branch line point ξ^0 are used to define the branch cuts of $\log(z_\alpha - \xi_\alpha)$ in the mapped planes.

where the circuit starts at I and ends at $F = I$. Since the logarithmic function is the only possible function that satisfies the force resultant and displacement jump along the prescribed circuit Γ_α in the z_α-plane, let

$$f_\alpha(z_\alpha) = \frac{C_\alpha}{2\pi i} \ln(z_\alpha - \xi_\alpha) \qquad (\alpha = 1, 2, 3), \tag{20}$$

where $\xi_\alpha = \eta_1 + p_\alpha \eta_2$ and C_α is determined below. Note that there is no sum on α. Substituting the complex potential function vector (20) into the displacement and stress function equations (18) we arrive at,

$$\begin{aligned} \phi_k &= 2\Re \sum_{\alpha=1}^{3} L_{k\alpha} \frac{C_\alpha}{2\pi i} \ln(z_\alpha - \xi_\alpha), \\ u_k &= 2\Re \sum_{\alpha=1}^{3} A_{k\alpha} \frac{C_\alpha}{2\pi i} \ln(z_\alpha - \xi_\alpha). \end{aligned} \tag{21}$$

Calculate the force resultant and the displacement jump around the circuit Γ and set them to $-\mathbf{r}$ and $\mathbf{b}$, respectively, to get

$$-r_k = -2\Re \left[\sum_{\alpha=1}^{3} L_{k\alpha} f_\alpha(z_\alpha) \right]_I^F = -2\Re \left[\sum_{\alpha=1}^{3} L_{k\alpha} C_\alpha \right] = -\sum_{\alpha=1}^{3} \left[L_{k\alpha} C_\alpha + \overline{L}_{k\alpha} \overline{C}_\alpha \right],$$

$$b_k \quad = \quad -2\Re\left[\sum_{\alpha=1}^{3} A_{k\alpha} f_\alpha(z_\alpha)\right]_I^F = 2\Re\left[\sum_{\alpha=1}^{3} A_{k\alpha} C_\alpha\right] = \sum_{\alpha=1}^{3}\left[L_{k\alpha}C_\alpha + \overline{L}_{k\alpha}\overline{C}_\alpha\right], \tag{22}$$

the component for of which is given by

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & \overline{A}_{11} & \overline{A}_{12} & \overline{A}_{13} \\ A_{21} & A_{22} & A_{23} & \overline{A}_{21} & \overline{A}_{22} & \overline{A}_{23} \\ A_{31} & A_{32} & A_{33} & \overline{A}_{31} & \overline{A}_{32} & \overline{A}_{33} \\ L_{11} & L_{12} & L_{13} & \overline{L}_{11} & \overline{L}_{12} & \overline{L}_{13} \\ L_{21} & L_{22} & L_{23} & \overline{L}_{21} & \overline{L}_{22} & \overline{L}_{23} \\ L_{31} & L_{32} & L_{33} & \overline{L}_{31} & \overline{L}_{32} & \overline{L}_{33} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ \overline{C}_1 \\ \overline{C}_2 \\ \overline{C}_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ r_1 \\ r_2 \\ r_3 \end{Bmatrix}. \tag{23}$$

Pre-multiplying the left and right sides of equation (23) by the matrix shown below to get

$$\begin{bmatrix} \mathbf{L^T} & \mathbf{A^T} \\ \overline{\mathbf{L}}^{\mathbf{T}} & \overline{\mathbf{A}}^{\mathbf{T}} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{L} & \overline{\mathbf{L}} \end{bmatrix} \begin{Bmatrix} \mathbf{C} \\ \overline{\mathbf{C}} \end{Bmatrix} = \begin{bmatrix} \mathbf{L^T} & \mathbf{A^T} \\ \overline{\mathbf{L}}^{\mathbf{T}} & \overline{\mathbf{A}}^{\mathbf{T}} \end{bmatrix} \begin{Bmatrix} \mathbf{b} \\ \mathbf{r} \end{Bmatrix}, \tag{24}$$

the expansion of which gives

$$\begin{bmatrix} \mathbf{L^T A} + \mathbf{A^T L} & \mathbf{L^T}\overline{\mathbf{A}} + \mathbf{A^T}\overline{\mathbf{L}} \\ \overline{\mathbf{L}}^{\mathbf{T}}\mathbf{A} + \overline{\mathbf{A}}^{\mathbf{T}}\mathbf{L} & \overline{\mathbf{L}}^{\mathbf{T}}\overline{\mathbf{A}} + \overline{\mathbf{A}}^{\mathbf{T}}\overline{\mathbf{L}} \end{bmatrix} \begin{Bmatrix} \mathbf{C} \\ \overline{\mathbf{C}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{L^T b} + \mathbf{A^T r} \\ \overline{\mathbf{L}}^{\mathbf{T}}\mathbf{b} + \overline{\mathbf{A}}^{\mathbf{T}}\mathbf{r} \end{Bmatrix}. \tag{25}$$

Since the matrix in the left hand side of (25) is a unit matrix, due to the orthogonality relations (17), this equation can be readily solved to get

$$C_\alpha = \sum_{k=1}^{3}(L_{k\alpha}b_k + A_{k\alpha}r_k) \qquad (\alpha = 1, 2, 3), \tag{26}$$

which can be substituted into (20) to obtain

$$f_\alpha(z_\alpha) = \frac{1}{2\pi i}\ln(z_\alpha - \xi_\alpha)\sum_{k=1}^{3}(L_{k\alpha}b_k + A_{k\alpha}r_k) \qquad (\alpha = 1, 2, 3). \tag{27}$$

It should be noted that there is no sum on α.

The displacement and stress contributions due to the line force and line dislocation are given by substituting (27) into (18) to get

$$u_i(z) = \Re\left\{\frac{1}{\pi i}\sum_{\alpha=1}^{3} A_{i\alpha}\ \ln(z_\alpha - \xi_\alpha)\sum_{k=1}^{3}(L_{k\alpha}b_k + A_{k\alpha}r_k)\right\} \tag{28}$$

and

$$\begin{aligned} \sigma_{1i}(z) \quad &= \quad -\Re\left\{\frac{1}{\pi i}\sum_{\alpha=1}^{3} p_\alpha L_{i\alpha}\ \frac{1}{z_\alpha - \xi_\alpha}\sum_{k=1}^{3}(L_{k\alpha}b_k + A_{k\alpha}r_k)\right\}, \\ \sigma_{2i}(z) \quad &= \quad \Re\left\{\frac{1}{\pi i}\sum_{\alpha=1}^{3} L_{i\alpha}\ \frac{1}{z_\alpha - \xi_\alpha}\sum_{k=1}^{3}(L_{k\alpha}b_k + A_{k\alpha}r_k)\right\}. \end{aligned} \tag{29}$$

The traction contribution is calculated by the stress formulas in (29) and the relation $t_i = \sigma_{ji} n_j$ to get

$$t_i(z) = \Re\left\{ \frac{1}{\pi i} \sum_{\alpha=1}^{3} (-p_\alpha n_1 + n_2) L_{i\alpha} \, \frac{1}{z_\alpha - \xi_\alpha} \sum_{k=1}^{3} (L_{k\alpha} b_k + A_{k\alpha} r_k) \right\}. \tag{30}$$

3.2 Explicit Formulas

3.2.1 Line Force

Consider a line force in x_k direction applied at (η_1, η_2). The the displacement component in the x_i direction at (x_1, x_2), denoted by $G_{ik}(x_1, x_2; \eta_1, \eta_2)$, is given from (28) by

$$G_{ik}(x_1, x_2; \eta_1, \eta_2) = \Im \frac{1}{\pi} \sum_{\alpha=1}^{3} A_{i\alpha} A_{k\alpha} \, \ln(z_\alpha - \xi_\alpha), \tag{31}$$

where $\Im$ indicates the imaginary part. The traction component in the x_i direction at (x_1, x_2), denoted by $H_{ik}(x_1, x_2; \eta_1, \eta_2)$, is given from (30) by

$$H_{ik}(x_1, x_2; \eta_1, \eta_2) = \Im \frac{1}{\pi} \sum_{\alpha=1}^{3} (-p_\alpha n_1 + n_2) L_{i\alpha} A_{k\alpha} \, \frac{1}{z_\alpha - \xi_\alpha}. \tag{32}$$

The corresponding stress component σ_{ij} at (x_1, x_2) is denoted by $S_{ijk}(x_1, x_2; \eta_1, \eta_2)$ and given, from (29), by

$$\begin{aligned} S_{1ik}(z) &= -\frac{1}{\pi} \Im \left\{ \sum_{\alpha=1}^{3} p_\alpha L_{i\alpha} A_{k\alpha} \, \frac{1}{z_\alpha - \xi_\alpha} \right\}, \\ S_{2ik}(z) &= \frac{1}{\pi} \Im \left\{ \sum_{\alpha=1}^{3} L_{i\alpha} A_{k\alpha} \, \frac{1}{z_\alpha - \xi_\alpha} \right\}. \end{aligned} \tag{33}$$

3.2.2 Line Dislocation

Consider a line dislocation at (η_1, η_2) with the unit Burgers vector in x_k direction. The resulting displacement component in the x_i direction at (x_1, x_2), denoted by $P_{ik}(x_1, x_2; \eta_1, \eta_2)$, is given from (28) by

$$P_{ik}(x_1, x_2; \eta_1, \eta_2) = \Im \frac{1}{\pi} \sum_{\alpha=1}^{3} A_{i\alpha} L_{k\alpha} \, \ln(z_\alpha - \xi_\alpha). \tag{34}$$

The traction component in the x_i direction at (x_1, x_2), denoted by $Q_{ik}(x_1, x_2; \eta_1, \eta_2)$, is given from (30) by

$$Q_{ik}(x_1, x_2; \eta_1, \eta_2) = \Im \frac{1}{\pi} \sum_{\alpha=1}^{3} (-p_\alpha n_1 + n_2) L_{i\alpha} L_{k\alpha} \, \frac{1}{z_\alpha - \xi_\alpha}. \tag{35}$$

The corresponding stress component σ_{ij} at (x_1, x_2) is denoted by $T_{ijk}(x_1, x_2; \eta_1, \eta_2)$ and given, from (29), by

$$\begin{aligned} T_{1ik}(x_1, x_2; \eta_1, \eta_2) &= -\frac{1}{\pi}\Im\left\{\sum_{\alpha=1}^{3} p_\alpha L_{i\alpha} L_{k\alpha}\, \frac{1}{z_\alpha - \xi_\alpha}\right\}, \\ T_{2ik}(x_1, x_2; \eta_1, \eta_2) &= \frac{1}{\pi}\Im\left\{\sum_{\alpha=1}^{3} L_{i\alpha} L_{k\alpha}\, \frac{1}{z_\alpha - \xi_\alpha}\right\}. \end{aligned} \tag{36}$$

4 Computer Codes

Variables used by the line force and dislocation are listed in Table 1. Subroutines for the line force and line dislocation are listed in Tables 2 and 3. FORTRAN 90 codes and sample input/output files accompany this paper.

4.1 Normalization used

A dimensional quantity q is normalized by its reference value q_0 to define its normalization $\tilde{q} = q/q_0$. We select the reference values for the stress and strain to be $\sigma_0 = 10^8$ (N/m^2) and $\epsilon_0 = 10^{-3}$. Other reference values are determined such that the normalized governing equations remain exactly the same form as the original equations. These are $s_0 = 10^{-11}$ (m^2/N) for the compliance and $u_0 = \epsilon_0 x_0$ (m) for the displacement, where x_0 is the characteristic length of the problem. All variables in the codes are normalized and they have magnitudes of order one. To recover the dimensional quantities just multiply the numerical results by the reference values. For example, $G_{ik} = u_0\tilde{G}_{ik}$ $(P_{ik} = u_0\tilde{P}_{ik})$, $H_{ik} = \sigma_0\tilde{H}_{ik}$ $(Q_{ik} = \sigma_0\bar{Q}_{ik})$ and $S_{oik} = \sigma_0\tilde{S}_{oik}$ $(T_{oik} = \sigma_0\tilde{T}_{oik})$, where quantities with tildes ($\tilde{}$) are numerical values obtained by the codes.

4.2 Branch of logarithmic functions

For the line dislocation, the branch line of $\ln(z-\xi)$ (in the physical plane) coincides with the slip line, which is assumed to be straight. Given the dislocation at the source point ξ (in the physical plane), the straight slip line is defined by specifying another arbitrary point, ξ^0 (called the branch line point), on this line as shown in Figure 1. The slip line is given by a line emanating from ξ and extending to and past ξ^0. The logarithmic function $\ln(z_\alpha - \xi_\alpha)$ in each of the mapped z_α-plane $(\alpha = 1, 2, 3)$ is defined as follows. First, get the image, ξ_α^0, of the branch line point ξ^0 (Figure 1). Next, calculate the principal value argument of $\xi_\alpha^0 - \xi_\alpha$, which is a line from the source ξ_α to the branch line point ξ_α^0 in the mapped z_α-plane. Set this value as the maximum, $\mathsf{upang(ia)}$, of the angular range so that

$$\mathsf{upang(ia)} - \pi < \arg(z_\alpha - \xi_\alpha) \le \mathsf{upang(ia)}, \tag{37}$$

where the minimum is given by upang(ia)$-\pi$. For the line force we use the principal value of the logarithm so that upang(ia) is always set to π and it is not necessary to specify ξ^0.

4.3 Tips

- For materials, such as isotropic solids, the characteristic roots are not distinct. Modify the compliance coefficients slightly to make these roots distinct.
- Use the compliance with the original dimension. The code will divide them by $10^{-11} m^2/N$ for normalization so that the actual values used in computation is of the order of one.
- Subroutine CHARACEQN uses MSIMSL (IMSL for COMPAC FORTRAN) to get characteristic roots of polynomial equations. Use other polynomial roots solver subroutine if MSIMSL is not available and please send the subroutine to the author (denda@jove.rutgers.edu) for future revision.

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Variable	Type	Dimension	Description
t	real	(21)	3D compliance
z	real	(2)	observation point
n	real	(2)	unit normal
xi	real	(2)	source point
xi0	real	(2)	branch cut point*
s	real	(15)	reduced compliance
p	complex	(3)	characteristic roots
a	complex	(3,3)	A matrix
l	complex	(3,3)	L matrix
za	complex	(3)	z_α
xia	complex	(3)	ξ_α
upang	real	(3)	upper angles of logarithmic functions*
g	real	(3,3)	G_{ik} (P_{ik} for line dislocation)
h	real	(3,3)	H_{ik} (Q_{ik} for line dislocation)
stress	real	(2,3,3)	S_{oik} (T_{oik} for line dislocation)

Table 1: FORTRAN variables used for line force and dislocation. Double precision is used for real and complex. *Line force subroutines do not use xio and upang.

Subroutine	Argument		Description
	In	Out	
Input	t, z, n, xi		Input 3D compliance & source/observation points data
REDCOMPLIANCE	t	s	Reduced compliance
CHARACEQN	s	p	Characteristic roots
LACOEF	s, p	l, a	L and A matrices
GENCOMPVAR	z, xi, p	za, xia	Generalized complex variables
GMAT	za, xia, a, l	g	G_{ik}
HMAT	za, n, xia, a, l, p	h	H_{ik}
SMAT	za, xia, a, l	stress	S_{oik}
WriteGHS		g, h, stress	Write results

Table 2: Subroutines for the line force.

Subroutine	Argument		Description
	In	Out	
Input	t, z, n, xi, xi0		Input 3D compliance & source/observation points data
REDCOMPLIANCE	t	s	Reduced compliance
CHARACEQN	s	p	Characteristic roots
LACOEF	s, p	l, a	L and A matrices
GENCOMPVAR	z, xi, xi0, p	za, xia, upang	Generalized complex variables
PMAT	za, xia, upang, a, l	g	P_{ik}
QMAT	za, n, xia, a, l, p	h	Q_{ik}
TMAT	za, xia, a, l	stress	T_{oik}
WriteGHS		g, h, stress	Write results

Table 3: Subroutines for line dislocation.