BEM Solutions of Frequency Domain Gradient Elastodynamic 3-D Problems

D. Polyzos[(a)], K. G. Tsepoura[(a)] and D. E. Beskos[(b)]
[(a)] Department of Mechanical Engineering and Aeronautics, University of Patras, Patras, GR26500, Greece
and
Institute of Chemical Engineering and High Temperature Chemical Processes-FORTH, Patras, GR26500, Greece
[(b)] Department of Civil Engineering, University of Patras, Patras, GR26500, Greece

Abstract

A boundary element methodology is presented for the frequency domain elastodynamic analysis of three-dimensional solids characterized by a linear elastic material behavior coupled with microstructural effects taken into account with the aid of the simple gradient elastic theory of Aifantis. A variational statement is established to determine all possible classical and non-classical (due to gradient terms) boundary conditions of the general boundary value problem. The gradient frequency domain elastodynamic fundamental solution is explicitly derived and used to construct the boundary integral representation of the solution with the aid of a reciprocal integral identity. In addition to a boundary integral representation for the displacement, a boundary integral representation for its normal derivative is also necessary for the complete formulation of a well posed problem. All the kernels in the integral equations are explicitly provided. Surface quadratic quadrilateral boundary elements are employed and the discretization is restricted only to the boundary. The solution procedure is described in detail. A numerical example serves to illustrate the method and demonstrate its accuracy. The present version of the method does not provide explicit expressions for the computation of interior stresses.

1. Introduction

In linear elastic materials with microstructure, such as polymers, polycrystals or granular materials, microstructural effects are important and have to be included in their constitutive equations. One way of successfully including these effects in a macroscopic manner is by using higher-order strain gradient theories.

Among those who have developed such theories one can mention Mindlin [1], [2], Aifantis and co-workers (Aifantis [3], Altan and Aifantis [4], Ru and Aifantis [5]) and Vardoulakis and co-workers (Vardoulakis and Sulem [6], Exadaktylos and Vardoulakis [7]). From the above theories, the most general and comprehensive is the one due to Mindlin [1], [2] involving 16 or in its special case 5 elastic constants, while the simplest is the one due to Aifantis (Aifantis [3], Altan and Aifantis [4], Ru and Aifantis [5]) involving only 3 elastic constants (two classical plus one non-classical). It can be easily proved that Aifantis' theory can be obtained as a special case of Mindlin's theory.

In recent years, a variety of boundary value problems of linear gradient elasticity have been solved analytically and the microstructural effects on the solution have been assessed under both static and dynamic conditions. The gradient elastic theories employed and the geometry of those problems were simple enough to permit an analytical solution to be obtained. One can mention here the works of Altan and Aifantis [4], Chang and Gao [15], Exadaktylos and Vardoulakis [10], Gutkin and Aifantis [9], Georgiadis et al. [16], Tsepoura et al. [11] and Papargyri et al. [12], [19] dealing with loaded half-spaces, fracture and dislocation mechanics and beams in tension or bending. It was found that use of gradient elasticity may lead to the elimination of singularities or discontinuities present in classical elasticity and the capturing of size effects and wave dispersion in cases where this was not possible in the classical elasticity context.

However, for realistic engineering problems characterised by complicated geometry and boundary conditions, analytical methods of solution are inadequate and resort has to be made to numerical methods, such as the finite element method (FEM) or the boundary element method (BEM). Among the efforts made for the FEM solution of boundary value problems in elastostatics in the framework of strain-gradient elastic behavior, one can mention the works of Shu et al. [17], Amanatidou and Aravas [18] and Teneketzis Tenek and Aifantis [22], all of them for the case of two-dimensions. The BEM has also been used for solving three-dimensional strain-gradient elastostatic problems by Tsepoura et al [14].

In this work the BEM in its direct form is employed for the solution of three-dimensional frequency domain elastodynamic problems in the framework of the strain-gradient theory due to Aifantis. The paper is organized as follows: Section 2 deals with the constitutive equations and the boundary conditions. The latter ones are obtained through a variational statement and comprise classical and non-classical ones. Section 3 presents the derivation of the fundamental solution of the problem, while section 4 presents the boundary integral representation of the gradient elastostatic problem. Section 5 describes the numerical implementation and solution procedure, which are illustrated by means of a numerical example.

2. Constitutive equations and boundary conditions

Consider a three dimensional (3-D) linear, gradient elastic body of volume V surrounded by a surface S, the geometry of which is described through a unit normal vector $\hat{\mathbf{n}}$ on S, and a Cartesian coordinate system $OX_1X_2X_3$ with its origin located interior to V. According to Mindlin's strain gradient theory [1], [2], the stored stain energy in V has the form

$$U = \int_V [\tilde{\boldsymbol{\tau}} : \tilde{\mathbf{e}} + (\tilde{\boldsymbol{\mu}})^{321} : \nabla\tilde{\mathbf{e}}]\,\mathrm{d}V = \int_V (\tau_{ij}e_{ij} + \mu_{ijk}\partial_i \delta e_{jk})\,\mathrm{d}V \tag{1}$$

where $\tilde{\boldsymbol{\tau}}$ is the classical second order elastic stress tensor being dual in energy to the strain elastic tensor $\tilde{\mathbf{e}}$, and $\tilde{\boldsymbol{\mu}}$ is the third order double stress tensor being dual in energy to the strain gradient $\nabla\tilde{\mathbf{e}}$. The double and triple dots in Eq. (1) indicate dyad and triad inner products, respectively, according to the rule

$$\begin{aligned}(\mathbf{a}\otimes\mathbf{b}) : (\mathbf{c}\otimes\mathbf{d}) &= (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d}) \\ (\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{m})\,\vdots\,(\mathbf{l}\otimes\mathbf{c}\otimes\mathbf{d}) &= (\mathbf{m}\cdot\mathbf{l})(\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d})\end{aligned} \tag{2}$$

where **a**, **b**, **c**, **d**, **m**, **l** are vectors in three dimensions, while ⊗ denotes dyadic product and the symbol $(\circ)^{321}$ means

$$(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c})^{321}=\mathbf{c}\otimes\mathbf{b}\otimes\mathbf{a} \tag{3}$$

It should be also mentioned that in connection with the subscripts of the double stress tensor μ_{ijk} appearing in Eq. (1), the first subscript indicates the direction of the normal vector on the surface on which double stresses act, while the other two subscripts have the same significance with the corresponding ones of the classical stress tensor τ_{ij}.

The dynamic governing equation of the considered gradient elastic body as well as the possible boundary conditions that establish a well-posed boundary value problem can be determined with the aid of the Hamilton's variational principle, written as

$$\int_{t_0}^{t_1}\delta\left(U-K\right)\mathrm{d}t-\int_{t_0}^{t_1}\delta W\mathrm{d}t=0 \tag{4}$$

where δ denotes variation, U is the strain energy given by Eq. (1), δW the variation of the work done by external forces and K the kinetic energy, which in terms of the displacement vector $\overline{\mathbf{u}}$ have the forms (Mindlin [1], [2])

$$\delta W=\int_V\overline{\mathbf{f}}\cdot\delta\mathbf{u}\,\mathrm{d}V+\int_S\overline{\mathbf{R}}\cdot[\hat{\mathbf{n}}\cdot\nabla(\delta\mathbf{u})]\,\mathrm{d}S+\int_S\overline{\mathbf{P}}\cdot\delta\mathbf{u}\,\mathrm{d}S+\sum_{C_a}\oint_{C_a}\left\{\overline{\mathbf{E}}\cdot\delta\mathbf{u}\right\}\mathrm{d}C \tag{5}$$

$$K=\frac{1}{2}\int_{\mathrm{V}}\rho\left|\dot{\overline{\mathbf{u}}}\right|^2\mathrm{d}V \tag{6}$$

In the above Eqs (5) and (6), $\overline{\mathbf{f}}$ denotes body forces, $\overline{\mathbf{P}}$ external surface tractions, $\overline{\mathbf{R}}$ surface double stresses, $\overline{\mathbf{E}}$ surface jump stresses and ρ mass density of the gradient elastic body, while an overdot indicates differentiation with respect to time.

Following a procedure similar to that described in Tsepoura et al. [14] for the static case and treating the variation of the kinetic energy as described in Papargyri-Beskoou et al. [19], the variational relation (4) leads to the equation of motion

$$\nabla\cdot\left(\tilde{\boldsymbol{\tau}}-\nabla\cdot\tilde{\boldsymbol{\mu}}\right)+\overline{\mathbf{f}}=\rho\,\ddot{\overline{\mathbf{u}}} \tag{7}$$

accompanied by the classical boundary conditions

$$\begin{aligned}\overline{\mathbf{P}}(\mathbf{x})=\hat{\mathbf{n}}\cdot\tilde{\boldsymbol{\tau}}-(\hat{\mathbf{n}}\otimes\hat{\mathbf{n}}):\frac{\partial\tilde{\boldsymbol{\mu}}}{\partial n}-\hat{\mathbf{n}}\cdot(\nabla_S\cdot\tilde{\boldsymbol{\mu}})-\hat{\mathbf{n}}\cdot[\nabla_S\cdot(\tilde{\boldsymbol{\mu}})^{213}]+\\ (\nabla_S\cdot\hat{\mathbf{n}})(\hat{\mathbf{n}}\otimes\hat{\mathbf{n}}):\tilde{\boldsymbol{\mu}}-(\nabla_S\hat{\mathbf{n}}):\tilde{\boldsymbol{\mu}}=\overline{\mathbf{P}}_0\end{aligned} \tag{8}$$

and/or

$$\overline{\mathbf{u}}=\overline{\mathbf{u}}_0$$

and the non-classical ones

$$\overline{\mathbf{R}}=\hat{\mathbf{n}}\cdot\tilde{\boldsymbol{\mu}}\cdot\hat{\mathbf{n}}=\overline{\mathbf{R}}_0\quad\text{and/or}\quad\frac{\partial\overline{\mathbf{u}}}{\partial n}=\overline{\mathbf{q}}_0 \tag{9}$$

$$\overline{\mathbf{E}}=\left\|(\hat{\mathbf{m}}\otimes\hat{\mathbf{n}}):\tilde{\boldsymbol{\mu}}\right\|=\overline{\mathbf{E}}_0 \tag{10}$$

with the initial conditions

$$\begin{aligned}\overline{\mathbf{u}}(\mathbf{x},t_0)&=\mathbf{U}(\mathbf{x})\\ \dot{\overline{\mathbf{u}}}(\mathbf{x},t_0)&=\mathbf{V}(\mathbf{x})\end{aligned} \tag{11}$$

In the above relations, ∇_S is the surface operator defined as $\nabla_S \equiv (\tilde{\mathbf{I}} - \hat{\mathbf{n}} \otimes \hat{\mathbf{n}}) \cdot \nabla$ with $\tilde{\mathbf{I}}$ being the unit tensor, and $\mathbf{P}_0, \overline{\mathbf{u}}_0, \overline{\mathbf{R}}_0, \overline{\mathbf{q}}_0, \overline{\mathbf{E}}_0, \mathbf{U}, \mathbf{V}$ denote prescribed values. It should be also mentioned that boundary condition (10) exists only when the boundary S is piecewise continuus containing a number of edge lines C_a , a =1, 2,…,n formed by the intersection of two surface portions $S_1^{(a)}$ and $S_2^{(a)}$ of S. In that case, $\hat{\mathbf{m}}$ represents the vector $\hat{\mathbf{m}} = \hat{\mathbf{s}} \times \hat{\mathbf{n}}$ with $\hat{\mathbf{s}}$ being the tangential vector to C_a, and the brackets $\|\cdot\|$ indicate that the enclosed quantity is the difference between the values on the surface portions $S_1^{(a)}$ and $S_2^{(a)}$.

Mindlin [2], considering isotropic materials and a special case of his theory where the macroscopic strain coincides to micro-deformation, proposed a modification of Hooke's law expressed by the following relations

$$\begin{aligned}
&\tilde{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\tau}} + \tilde{\mathbf{s}} \\
&\tilde{\boldsymbol{\tau}} = 2\mu \tilde{\mathbf{e}} + \lambda (\nabla \cdot \overline{\mathbf{u}}) \tilde{\mathbf{I}} \\
&\tilde{\mathbf{e}} = (\nabla \overline{\mathbf{u}} + \overline{\mathbf{u}} \nabla)/2 \\
&\tilde{\mathbf{s}} = -\left[2\mu c_3 \nabla^2 \tilde{\mathbf{e}} + \lambda c_1 \tilde{\mathbf{I}} \nabla^2 (\nabla \cdot \overline{\mathbf{u}}) + \lambda c_2 \nabla\nabla (\nabla \cdot \overline{\mathbf{u}}) \right]
\end{aligned} \tag{12}$$

where ∇^2 is the Laplacian, $\tilde{\boldsymbol{\sigma}}$ is the total stress tensor, $\tilde{\boldsymbol{\tau}}$ and $\tilde{\mathbf{s}}$ are the so-called by Mindlin, Cauchy stress tensor and relative stress tensor, respectively, and $\tilde{\mathbf{e}}$ is the strain tensor. The total stresses are correlateds to strains and strain gradients through five independent material constants, i.e., λ, μ, c_1, c_2 and c_3 with the first two being the well known Lame' constants.

A simpler and mathematically more tractable constitutive equation is that proposed by Aifantis and co-workers (Altan and Aifantis [4], Ru and Aifantis [5]) and correlates the relative stress tensor $\tilde{\mathbf{s}}$ with the double stresses $\tilde{\boldsymbol{\mu}}$ according to the relations

$$\begin{aligned}
&\tilde{\mathbf{s}} = -\nabla \cdot \tilde{\boldsymbol{\mu}} \\
&\tilde{\boldsymbol{\mu}} = g^2 \nabla \tilde{\boldsymbol{\tau}}
\end{aligned} \tag{13}$$

where g^2 is the volumetric strain gradient energy coefficient, the only constant which relates the microstructure with the macrostructure. It is easy to see that this simple theory can be obtained as a special case of that of Mindlin if one sets $c_1 = c_2 = g^2$ and $c_3 = 0$ in Eqs (12).

Adopting the above simple theory of Aifantis and inserting the constitutive Eqs (13) into Eq. (7) one obtains the following equation of motion of a gradient elastic continuum in terms of the displacement field $\overline{\mathbf{u}}$:

$$\mu \nabla^2 \overline{\mathbf{u}} + (\lambda + \mu) \nabla\nabla \cdot \overline{\mathbf{u}} - g^2 \nabla^2 \left(\mu \nabla^2 \overline{\mathbf{u}} + (\lambda + \mu) \nabla\nabla \cdot \overline{\mathbf{u}} \right) + \overline{\mathbf{f}} = \rho \ddot{\overline{\mathbf{u}}} \tag{14}$$

Considering harmonic with respect to time t behavior for both displacements and body forces $\left(\overline{\mathbf{u}} = \mathbf{u} e^{-i\omega t}, \overline{\mathbf{f}} = \mathbf{f} e^{-i\omega t} \right)$, Eq. (14) is transformed to a frequency domain equation of motion of the form

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla\nabla \cdot \mathbf{u} - g^2 \nabla^2 \left(\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla\nabla \cdot \mathbf{u} \right) + \mathbf{f} + \rho \omega^2 \mathbf{u} = 0 \tag{15}$$

with ω being the excitation frequency.

3. Frequency domain gradient elastic fundamental solution

In this section the frequency domain fundamental solution of an infinitely extended gradient elastic material, the dynamic behavior of which is described by Eq. (15), is explicitly derived. This fundamental solution is defined as the solution of the partial differential equation

$$\Im\tilde{\mathbf{U}}^*(r) = \delta(\mathbf{x}-\mathbf{y})\tilde{\mathbf{I}} \tag{16}$$

where δ is the Dirac δ-function, **x** is the point where the displacement field $\tilde{\mathbf{U}}^*$ due to a unit force applied at a point **y** should be obtained, $r = |\mathbf{x}-\mathbf{y}|$ and $\Im$ is the linear operator

$$\Im \equiv \mu\nabla^2 + (\lambda+\mu)\nabla\nabla\cdot - g^2\nabla^2\left(\mu\nabla^2 + (\lambda+\mu)\nabla\nabla\cdot\right) + \rho\omega^2 \tag{17}$$

According to the Helmholtz decomposition applied to dyadic fields (Dassios and Lindel [20]), the fundamental solution $\tilde{\mathbf{U}}^*(r)$ can be decomposed into irrotational and solenoidal parts as

$$\tilde{\mathbf{u}}^*(r) = \nabla\nabla\varphi(r) + \nabla\nabla\times\mathbf{A}(r) + \nabla\times\nabla\times\tilde{\mathbf{G}}(r) \tag{18}$$

where $\varphi(r)$ is a scalar function, $\mathbf{A}(r)$ a vector function and $\tilde{\mathbf{G}}(r)$ a dyadic function.
Since vector $\mathbf{A}(r)$ is a function of r, it is easy to see that

$$\nabla\times\mathbf{A}(r) = 0 \tag{19}$$

Substituting Eqs (18) and (19) into Eq. (16) and taking into account the relation

$$\nabla^2\left(\frac{1}{4\pi r}\right) = -\delta(\mathbf{r}) \tag{20}$$

and the identity

$$\nabla^2 = \nabla\text{div} - \nabla\times\text{rot} \tag{21}$$

Eq. (16) takes the form

$$\begin{aligned}&\nabla\nabla\left[(\lambda+2\mu)\left[\nabla^2\varphi(r) - g^2\nabla^4\varphi(r)\right] + \rho\omega^2\varphi(r)\right] + \\ &\nabla\times\nabla\times\left[\mu\left(\nabla^2\tilde{\mathbf{G}}(r) - g^2\nabla^4\tilde{\mathbf{G}}(r)\right) + \rho\omega^2\tilde{\mathbf{G}}(r)\right] = \\ &\nabla\nabla\left(\frac{1}{4\pi r}\right) - \nabla\times\nabla\times\left(\frac{1}{4\pi r}\tilde{\mathbf{I}}\right)\end{aligned} \tag{22}$$

Due to the irrotational and solenoidal nature of $\varphi(r)$ and $\tilde{\mathbf{G}}(r)$, respectively, Eq. (22) is satisfied when

$$\nabla^2\varphi(r) - g^2\nabla^4\varphi(r) + k_p^2\,\varphi(r) = \frac{1}{4\pi\,(\lambda+2\mu)\,r} \tag{23}$$

$$\nabla^2\tilde{\mathbf{G}}(r) - g^2\nabla^4\tilde{\mathbf{G}}(r) + k_s^2\,\tilde{\mathbf{G}}(r) = -\frac{1}{4\pi\,\mu\,r}\tilde{\mathbf{I}} \tag{24}$$

where $k_p = \sqrt{\rho\omega^2/(\lambda+2\mu)}$ and $k_s = \sqrt{\rho\omega^2/\mu}$ stand for the wave numbers of classical longitudinal and shear waves, respectively.
Utilizing the dispersion relations of the homogeneous Eqs (23) and (24), i.e.,

$$\begin{aligned}k_1^2\left(1+g^2k_1^2\right) &= k_p^2 \\ k_2^2\left(1+g^2k_2^2\right) &= k_s^2\end{aligned} \tag{25}$$

Eqs (23) and (24) become

$$\nabla^2\varphi(r) - g^2\nabla^4\varphi(r) + k_1^2\left(1 + g^2k_1^2\right)\varphi(r) = \frac{1}{4\pi\,(\lambda + 2\mu)\,r} \tag{26}$$

$$\nabla^2\widetilde{\mathbf{G}}(r) - g^2\nabla^4\widetilde{\mathbf{G}}(r) + k_2^2\left(1 + g^2k_2^2\right)\widetilde{\mathbf{G}}(r) = -\frac{1}{4\pi\,\mu\,r}\widetilde{\mathbf{I}} \tag{27}$$

The scalar $\varphi(r)$ and tensor $\widetilde{\mathbf{G}}(r)$ functions, which satisfy Eqs (26) and (27), respectively, have the form

$$\varphi(r) = \frac{1}{4\pi\rho\omega^2}\left(\frac{1}{r} - \frac{1 + g^2k_1{}^2}{1 + 2g^2k_1{}^2}\frac{e^{-ik_1r}}{r} - \frac{g^2k_1{}^2}{1 + 2g^2k_1{}^2}\frac{e^{-\left(\sqrt{\frac{1}{g^2} + k_1{}^2}\right)r}}{r}\right) \tag{28}$$

$$\widetilde{\mathbf{G}}(r) = -\frac{1}{4\pi\rho\omega^2}\left(\frac{1}{r} - \frac{1 + g^2k_2{}^2}{1 + 2g^2k_2{}^2}\frac{e^{-ik_2r}}{r} - \frac{g^2k_2{}^2}{1 + 2g^2k_2{}^2}\frac{e^{-\left(\sqrt{\frac{1}{g^2} + k_2{}^2}\right)r}}{r}\right)\widetilde{\mathbf{I}} \tag{29}$$

where $i = \sqrt{-1}$.

Inserting Eqs (28) and (29) into Eq. (18) and taking into account Eq. (19), the fundamental solution of Eq. (16) takes the final form

$$\widetilde{\mathbf{U}}^*(r) = \frac{1}{16\pi\mu(1 - \nu)}\left[\Psi(r,g)\widetilde{\mathbf{I}} - \mathrm{X}(r,g)\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}\right] \tag{30}$$

where $\hat{\mathbf{r}}$ the unit vector in the direction $\mathbf{r} = \mathbf{y} - \mathbf{x}$ and X and Ψ are scalar functions given by the relations

$$\begin{aligned}\mathrm{X}(r,g) = \frac{4(1-\nu)}{\left(1 + g^2k_2{}^2\right)}\Bigg\{\Bigg[&\frac{1 + g^2k_2^2}{1 + 2g^2k_2^2}\left(1 + \frac{3}{ik_2r} - \frac{3}{k_2^2\,r^2}\right)\frac{e^{-ik_2r}}{r} - \\ &\frac{1 + g^2k_1^2}{1 + 2g^2k_1^2}\left(\frac{k_1^2}{k_2^2}\right)\left(1 + \frac{3}{ik_1r} - \frac{3}{k_1^2\,r^2}\right)\frac{e^{-ik_1r}}{r} + \\ &\frac{1 + g^2k_1^2}{1 + 2g^2k_1^2}\left(\frac{k_1^2}{k_2{}^2}\right)\left(1 + \frac{3}{\left(\sqrt{\frac{1}{g^2} + k_1^2}\right)r} + \frac{3}{\left(\frac{1}{g^2} + k_1^2\right)r^2}\right)\frac{e^{-\left(\sqrt{\frac{1}{g^2} + k_1^2}\right)r}}{r} - \\ &\frac{1 + g^2k_2^2}{1 + 2g^2k_2^2}\left(1 + \frac{3}{\left(\sqrt{\frac{1}{g^2} + k_2^2}\right)r} + \frac{3}{\left(\frac{1}{g^2} + k_2^2\right)r^2}\right)\frac{e^{-\left(\sqrt{\frac{1}{g^2} + k_2^2}\right)r}}{r}\Bigg]\Bigg\}\end{aligned} \tag{31}$$

$$\Psi(r,g) = \frac{4(1-\nu)}{\left(1+g^2k_2^{\,2}\right)}\left\{\left[\frac{1+g^2k_2^2}{1+2g^2k_2^2}\left(1+\frac{1}{ik_2r}-\frac{1}{k_2^2\,r^2}\right)\frac{e^{-ik_2r}}{r} - \right.\right.$$
$$\frac{1+g^2k_1^2}{1+2g^2k_1^2}\left(\frac{k_1^2}{k_2^2}\right)\left(\frac{1}{ik_1r}-\frac{1}{k_1^2\,r^2}\right)\frac{e^{-ik_1r}}{r} +$$
$$\frac{1+g^2k_1^2}{1+2g^2k_1^2}\left(\frac{k_1^2}{k_2^{\,2}}\right)\left(\frac{1}{\left(\sqrt{\frac{1}{g^2}+k_1^2}\right)r}+\frac{1}{\left(\frac{1}{g^2}+k_1^2\right)r^2}\right)\frac{e^{-\left(\sqrt{\frac{1}{g^2}+k_1^2}\right)r}}{r} -$$
$$\left.\left.\frac{1+g^2k_2^2}{1+2g^2k_2^2}\left(1+\frac{1}{\left(\sqrt{\frac{1}{g^2}+k_2^2}\right)r}+\frac{1}{\left(\frac{1}{g^2}+k_2^2\right)r^2}\right)\frac{e^{-\left(\sqrt{\frac{1}{g^2}+k_2^2}\right)r}}{r}\right]\right\} \tag{32}$$

4. Boundary integral representation of the problem

In this section the boundary integral representation of a gradient elastodynamic problem, for the most general case of a non-smooth boundary, is derived in the frequency domain. Consider a finite 3-D gradient elastic body of volume V surrounded by a surface S consisting, for the shake of simplicity, of two smooth surfaces S_1 and S_2 intersecting across the closed line C, whose motion in the frequency domain is described by Eq. (15). Assuming two deformation states of the same body $(\mathbf{f},\mathbf{u},\boldsymbol{\sigma})$ and $(\mathbf{f}^*,\mathbf{u}^*,\boldsymbol{\sigma}^*)$ and following a similar procedure to that described in Tsepoura et al. [14], one can obtain the following reciprocal identity valid for the considered gradient elastic continuum:

$$\int_V \left\{\mathbf{f}^*\cdot\mathbf{u}-\mathbf{f}\cdot\mathbf{u}^*\right\}\mathrm{d}V + \int_S \left\{\mathbf{P}^*\cdot\mathbf{u}-\mathbf{P}\cdot\mathbf{u}^*\right\}\mathrm{d}S = \int_S \left\{\mathbf{R}\cdot\frac{\partial\mathbf{u}^*}{\partial n} - \mathbf{R}^*\cdot\frac{\partial\mathbf{u}}{\partial n}\right\}\mathrm{d}S + \sum_{C_a}\oint_{C_a}\left\{\mathbf{E}\cdot\mathbf{u}^*-\mathbf{E}^*\cdot\mathbf{u}\right\}\mathrm{d}C \tag{33}$$

where the surface tractions $\mathbf{P}$, $\mathbf{R}$ and $\mathbf{E}$ have the form

$$\mathbf{P} = \hat{\mathbf{n}}\cdot\tilde{\boldsymbol{\tau}} - (\hat{\mathbf{n}}\otimes\hat{\mathbf{n}}):\frac{\partial\tilde{\boldsymbol{\mu}}}{\partial n} - \hat{\mathbf{n}}\cdot(\nabla_S\cdot\tilde{\boldsymbol{\mu}}) - \hat{\mathbf{n}}\cdot[\nabla_S\cdot(\tilde{\boldsymbol{\mu}})^{213}] + (\nabla_S\cdot\hat{\mathbf{n}})(\hat{\mathbf{n}}\otimes\hat{\mathbf{n}}):\tilde{\boldsymbol{\mu}} - (\nabla_S\hat{\mathbf{n}}):\tilde{\boldsymbol{\mu}}$$
$$\mathbf{R} = \hat{\mathbf{n}}\cdot\tilde{\boldsymbol{\mu}}\cdot\hat{\mathbf{n}} \tag{34}$$
$$\mathbf{E} = \left\|(\hat{\mathbf{m}}\otimes\hat{\mathbf{n}}):\tilde{\boldsymbol{\mu}}\right\|$$

Assume that the displacement field $\widetilde{\mathbf{U}}^*$, appearing in the reciprocal identity (33), is the result of a body force having the form

$$\mathbf{f}^*(\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})\hat{\mathbf{e}} \tag{35}$$

with δ being the Dirac δ-function and $\hat{\mathbf{e}}$ the direction of a unit force acting at point **y**. Recalling the definition of the fundamental solution derived in section 3, it is easy to see that the displacement field $\mathbf{U}^*$ can be represented by means of the fundamental displacement tensor $\widetilde{\mathbf{U}}^*(\mathbf{x},\mathbf{y})$ given by Eq. (30), according to the relation

$$\mathbf{U}^*(\mathbf{y}) = \widetilde{\mathbf{U}}^*(\mathbf{x},\mathbf{y})\cdot\hat{\mathbf{e}} \tag{36}$$

Inserting the above expression of $\mathbf{U}^*$ in (32) and assuming zero body forces **f**=**0**, one obtains

$$\begin{aligned}
&\int_V \{\delta(\mathbf{x}-\mathbf{y})\hat{\mathbf{e}}\cdot\mathbf{u}(\mathbf{y})\}\mathrm{d}V_{\mathrm{y}} + \\
&\int_S \{[\widetilde{\mathbf{P}}^*(\mathbf{x},\mathbf{y})\cdot\hat{\mathbf{e}}]\cdot\mathbf{u}(\mathbf{y}) - \mathbf{P}(\mathbf{y})\cdot[\widetilde{\mathbf{U}}^*(\mathbf{x},\mathbf{y})\cdot\hat{\mathbf{e}}]\}\mathrm{d}S_{\mathrm{y}} = \\
&\int_S \left\{\mathbf{R}(\mathbf{y})\cdot[\frac{\partial\widetilde{\mathbf{U}}^*(\mathbf{x},\mathbf{y})}{\partial n_{\mathrm{y}}}\cdot\hat{\mathbf{e}}] - [\widetilde{\mathbf{R}}^*(\mathbf{x},\mathbf{y})\cdot\hat{\mathbf{e}}]\cdot\frac{\partial\mathbf{u}(\mathbf{y})}{\partial n_{\mathrm{y}}}\right\}\mathrm{d}S_{\mathrm{y}} + \\
&\oint_C \{\mathbf{E}(\mathbf{y})\cdot[\widetilde{\mathbf{U}}^*(\mathbf{x},\mathbf{y})\cdot\hat{\mathbf{e}}] - [\widetilde{\mathbf{E}}^*(\mathbf{x},\mathbf{y})\cdot\hat{\mathbf{e}}]\cdot\mathbf{u}(\mathbf{y})\}\mathrm{d}C_{\mathrm{y}}
\end{aligned} \tag{37}$$

or

$$\begin{aligned}
&(\int_V \{\delta(\mathbf{x}-\mathbf{y})\mathbf{u}(\mathbf{y})\}\mathrm{d}V_{\mathrm{y}})\cdot\hat{\mathbf{e}} + \\
&(\int_S \{[\widetilde{\mathbf{P}}^*(\mathbf{x},\mathbf{y})]^{\mathrm{T}}\cdot\mathbf{u}(\mathbf{y}) - \mathbf{P}(\mathbf{y})\cdot\widetilde{\mathbf{U}}^*(\mathbf{x},\mathbf{y})\}\mathrm{d}S_{\mathrm{y}})\cdot\hat{\mathbf{e}} = \\
&(\int_S \left\{\left(\frac{\partial\widetilde{\mathbf{U}}^*(\mathbf{x},\mathbf{y})}{\partial n_{\mathrm{y}}}\right)^{\mathrm{T}}\cdot\mathbf{R}(\mathbf{y}) - [\widetilde{\mathbf{R}}^*(\mathbf{x},\mathbf{y})]^T\cdot\frac{\partial\mathbf{u}(\mathbf{y})}{\partial n_{\mathrm{y}}}\right\}\mathrm{d}S_{\mathrm{y}})\cdot\hat{\mathbf{e}} + \\
&(\oint_C \{\mathbf{E}(\mathbf{y})\cdot\widetilde{\mathbf{U}}^*(\mathbf{x},\mathbf{y}) - [\widetilde{\mathbf{E}}^*(\mathbf{x},\mathbf{y})]^{\mathrm{T}}\cdot\mathbf{u}(\mathbf{y})\}\mathrm{d}C_{\mathrm{y}})\cdot\hat{\mathbf{e}}
\end{aligned} \tag{38}$$

Considering that relation (38) is valid for any direction $\hat{\mathbf{e}}$ and taking into account the symmetry of the fundamental displacement $\widetilde{\mathbf{u}}^*$, one obtains the boundary integral equation

$$\begin{aligned}
&\widetilde{\mathbf{c}}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x}) + \int_S \{[\widetilde{\mathbf{P}}^*(\mathbf{x},\mathbf{y})]^{\mathrm{T}}\cdot\mathbf{u}(\mathbf{y}) - \widetilde{\mathbf{U}}^*(\mathbf{x},\mathbf{y})\cdot\mathbf{P}(\mathbf{y})\}\mathrm{d}S_{\mathrm{y}} = \\
&\int_S \left\{\left(\frac{\partial\widetilde{\mathbf{U}}^*(\mathbf{x},\mathbf{y})}{\partial n_{\mathrm{y}}}\right)^{\mathrm{T}}\cdot\mathbf{R}(\mathbf{y}) - [\widetilde{\mathbf{R}}^*(\mathbf{x},\mathbf{y})]^{\mathrm{T}}\cdot\frac{\partial\mathbf{u}(\mathbf{y})}{\partial n_{\mathrm{y}}}\right\}\mathrm{d}S_{\mathrm{y}} + \\
&\oint_C \{\widetilde{\mathbf{U}}^*(\mathbf{x},\mathbf{y})\cdot\mathbf{E}(\mathbf{y}) - [\widetilde{\mathbf{E}}^*(\mathbf{x},\mathbf{y})]^{\mathrm{T}}\cdot\mathbf{u}(\mathbf{y})\}\mathrm{d}C_{\mathrm{y}}
\end{aligned} \tag{39}$$

where $\tilde{\mathbf{c}}(\mathbf{x})$ is the well known jump-tensor of classical boundary integral representation (Dominguez [21]). Utilizing the symbols $\tilde{\tilde{\mathbf{P}}}^*$, $\tilde{\tilde{\mathbf{Q}}}^*$, $\tilde{\tilde{\mathbf{R}}}^*$ and $\tilde{\tilde{\mathbf{E}}}^*$ instead of $\left(\tilde{\mathbf{P}}^*\right)^{\mathrm{T}}$, $\left(\dfrac{\partial \tilde{\mathbf{U}}^*}{\partial n}\right)^{\mathrm{T}}$, $\left(\tilde{\mathbf{R}}^*\right)^{\mathrm{T}}$ and $\left(\tilde{\mathbf{E}}^*\right)^{\mathrm{T}}$, respectively, Eq. (39) receives the form

$$\begin{aligned}
&\tilde{\mathbf{c}}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x}) + \int_S \left\{\tilde{\tilde{\mathbf{P}}}^*(\mathbf{x},\mathbf{y})\cdot\mathbf{u}(\mathbf{y}) - \tilde{\mathbf{U}}^*(\mathbf{x},\mathbf{y})\cdot\mathbf{P}(\mathbf{y})\right\}\mathrm{d}S_{\mathrm{y}} = \\
&\int_S \left\{\tilde{\tilde{\mathbf{Q}}}^*(\mathbf{x},\mathbf{y})\cdot\mathbf{R}(\mathbf{y}) - \tilde{\tilde{\mathbf{R}}}^*(\mathbf{x},\mathbf{y})\cdot\frac{\partial\mathbf{u}(\mathbf{y})}{\partial n_{\mathrm{y}}}\right\}\mathrm{d}S_{\mathrm{y}} + \\
&\oint_C \left\{\tilde{\mathbf{U}}^*(\mathbf{x},\mathbf{y})\cdot\mathbf{E}(\mathbf{y}) - \tilde{\tilde{\mathbf{E}}}^*(\mathbf{x},\mathbf{y})\cdot\mathbf{u}(\mathbf{y})\right\}\mathrm{d}C_{\mathrm{y}}
\end{aligned} \tag{40}$$

In case the boundary S is smooth, then integral equation (40) is reduced to

$$\begin{aligned}
&\frac{1}{2}\mathbf{u}(\mathbf{x}) + \int_S \left\{\tilde{\tilde{\mathbf{P}}}^*(\mathbf{x},\mathbf{y})\cdot\mathbf{u}(\mathbf{y}) - \tilde{\mathbf{U}}^*(\mathbf{x},\mathbf{y})\cdot\mathbf{P}(\mathbf{y})\right\}\mathrm{d}S_{\mathrm{y}} = \\
&\int_S \left\{\tilde{\tilde{\mathbf{Q}}}^*(\mathbf{x},\mathbf{y})\cdot\mathbf{R}(\mathbf{y}) - \tilde{\tilde{\mathbf{R}}}^*(\mathbf{x},\mathbf{y})\cdot\frac{\partial\mathbf{u}(\mathbf{y})}{\partial n_{\mathrm{y}}}\right\}\mathrm{d}S_{\mathrm{y}}
\end{aligned} \tag{41}$$

All the kernels appearing in the integral equations (40) and (41) are given explicitly in Appendix I.

Observing Eq. (40), one easily realizes that this equation contains two unknown vector fields, $\mathbf{u}(\mathbf{x})$ and $\partial\mathbf{u}(\mathbf{x})/\partial n$. For example, for the case of the traction field $\mathbf{P}(\mathbf{x})$ prescribed on S (classical boundary condition) as well as the fields $\mathbf{R}(\mathbf{x})$ and $\mathbf{E}(\mathbf{x})$ prescribed on S (non-classical boundary condition), the unknown vector fields in (40) are two, $\mathbf{u}(\mathbf{x})$ and $\partial\mathbf{u}(\mathbf{x})/\partial n$. Thus, the evaluation of the unknown fields $\mathbf{u}(\mathbf{x})$ and $\partial\mathbf{u}(\mathbf{x})/\partial n$ requires the existence of one more integral equation. This integral equation is obtained by applying the operator $\partial/\partial n_{\mathbf{x}}$ on (40) and has the form

$$\begin{aligned}
&\tilde{\mathbf{c}}(\mathbf{x})\cdot\frac{\partial\mathbf{u}(\mathbf{x})}{\partial n_{\mathrm{x}}} + \int_S \left\{\frac{\partial\tilde{\tilde{\mathbf{P}}}^*(\mathbf{x},\mathbf{y})}{\partial n_{\mathrm{x}}}\cdot\mathbf{u}(\mathbf{y}) - \frac{\partial\tilde{\mathbf{U}}^*(\mathbf{x},\mathbf{y})}{\partial n_{\mathrm{x}}}\cdot\mathbf{p}(\mathbf{y})\right\}\mathrm{d}S_{\mathrm{y}} = \\
&\int_S \left\{\frac{\partial\tilde{\tilde{\mathbf{Q}}}^*(\mathbf{x},\mathbf{y})}{\partial n_{\mathrm{x}}}\cdot\mathbf{R}(\mathbf{y}) - \frac{\partial\tilde{\tilde{\mathbf{R}}}^*(\mathbf{x},\mathbf{y})}{\partial n_{\mathrm{x}}}\cdot\frac{\partial\mathbf{u}(\mathbf{y})}{\partial n_{\mathrm{y}}}\right\}\mathrm{d}S_{\mathrm{y}} + \\
&\oint_C \left\{\frac{\partial\tilde{\mathbf{U}}^*(\mathbf{x},\mathbf{y})}{\partial n_{\mathrm{x}}}\cdot\mathbf{E}(\mathbf{y}) - \frac{\partial\tilde{\tilde{\mathbf{E}}}^*(\mathbf{x},\mathbf{y})}{\partial n_{\mathrm{x}}}\cdot\mathbf{u}(\mathbf{y})\right\}\mathrm{d}C_{\mathrm{y}}
\end{aligned} \tag{42}$$

For smooth boundaries S, integral equation (42) is reduced to

$$\frac{1}{2}\frac{\partial \mathbf{u}(\mathbf{x})}{\partial n_{\mathbf{x}}}+\int_S\left\{\frac{\partial \widetilde{\widetilde{\mathbf{P}}}^*(\mathbf{x},\mathbf{y})}{\partial n_{\mathbf{x}}}\cdot\mathbf{u}(\mathbf{y})-\frac{\partial \widetilde{\mathbf{U}}^*(\mathbf{x},\mathbf{y})}{\partial n_{\mathbf{x}}}\cdot\mathbf{p}(\mathbf{y})\right\}\mathrm{d}S_{\mathbf{y}}= \int_S\left\{\frac{\partial \widetilde{\widetilde{\mathbf{Q}}}^*(\mathbf{x},\mathbf{y})}{\partial n_{\mathbf{x}}}\cdot\mathbf{R}(\mathbf{y})-\frac{\partial \widetilde{\widetilde{\mathbf{R}}}^*(\mathbf{x},\mathbf{y})}{\partial n_{\mathbf{x}}}\cdot\frac{\partial \mathbf{u}(\mathbf{y})}{\partial n_{\mathbf{y}}}\right\}\mathrm{d}S_{\mathbf{y}} \qquad (43)$$

The kernels appearing in Eqs (42) and (43) are given explicitly in Appendix II.

The integral equations (40) and (42) accompanied by the classical and non-classical boundary conditions (8), (9) and (10), respectively, form the integral representation of any gradient elastic boundary value problem satisfying the partial differential equation (15). The present version of the formulation is restricted to problems requiring computation of tractions and stresses only at the boundary. Work is under way to produce expressions for the explicit computation of interior stresses.

5. Numerical implementation and numerical results

In this section, the boundary element formulation and solution procedure of a 3-D static gradient elastic problem described in integral form by Eqs (41) and (43) is presented in detail. The goal of the boundary element methodology is to solve numerically the well-posed boundary value problem consisting of the system of the two integral equations (41) and (43) and the boundary conditions (8) and (9). To this end, the smooth surface S is discretized into B eight-noded quadrilateral and/or six-noded triangular quadratic continuous isoparametric boundary elements. For a nodal point k, the discretized integral equations (41) and (43) have the form

$$\begin{aligned}&\frac{1}{2}\mathbf{u}(\mathbf{x}^k)+\sum_{e=1}^{B}\sum_{a=1}^{A(e)}\int_{-1}^{1}\int_{-1}^{1}\widetilde{\widetilde{\mathbf{P}}}^*\left(\mathbf{x}^k,\mathbf{y}^e(\xi_1,\xi_2)\right)N^a(\xi_1,\xi_2)J(\xi_1,\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2\cdot\mathbf{u}_a^e+\\&\sum_{e=1}^{B}\sum_{a=1}^{A(e)}\int_{-1}^{1}\int_{-1}^{1}\widetilde{\widetilde{\mathbf{R}}}^*\left(\mathbf{x}^k,\mathbf{y}^e(\xi_1,\xi_2)\right)N^a(\xi_1,\xi_2)J(\xi_1,\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2\cdot\mathbf{q}_a^e=\\&\sum_{e=1}^{B}\sum_{a=1}^{A(e)}\int_{-1}^{1}\int_{-1}^{1}\widetilde{\mathbf{U}}^*\left(\mathbf{x}^k,\mathbf{y}^e(\xi_1,\xi_2)\right)N^a(\xi_1,\xi_2)J(\xi_1,\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2\cdot\mathbf{P}_a^e+\\&\sum_{e=1}^{B}\sum_{a=1}^{A(e)}\int_{-1}^{1}\int_{-1}^{1}\widetilde{\widetilde{\mathbf{Q}}}^*\left(\mathbf{x}^k,\mathbf{y}^e(\xi_1,\xi_2)\right)N^a(\xi_1,\xi_2)J(\xi_1,\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2\cdot\mathbf{R}_a^e\end{aligned} \qquad (44)$$

$$\begin{aligned}&\frac{1}{2}\mathbf{q}(\mathbf{x}^k)+\sum_{e=1}^{B}\sum_{a=1}^{A(e)}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial\widetilde{\widetilde{\mathbf{P}}}^*\left(\mathbf{x}^k,\mathbf{y}^e(\xi_1,\xi_2)\right)}{\partial n_{\mathbf{x}}}N^a(\xi_1,\xi_2)J(\xi_1,\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2\cdot\mathbf{u}_a^e+\\&\sum_{e=1}^{B}\sum_{a=1}^{A(e)}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial\widetilde{\widetilde{\mathbf{R}}}^*\left(\mathbf{x}^k,\mathbf{y}^e(\xi_1,\xi_2)\right)}{\partial n_{\mathbf{x}}}N^a(\xi_1,\xi_2)J(\xi_1,\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2\cdot\mathbf{q}_a^e=\\&\sum_{e=1}^{B}\sum_{a=1}^{A(e)}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial\widetilde{\mathbf{U}}^*\left(\mathbf{x}^k,\mathbf{y}^e(\xi_1,\xi_2)\right)}{\partial n_{\mathbf{x}}}N^a(\xi_1,\xi_2)J(\xi_1,\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2\cdot\mathbf{P}_a^e+\\&\sum_{e=1}^{B}\sum_{a=1}^{A(e)}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial\widetilde{\widetilde{\mathbf{Q}}}^*\left(\mathbf{x}^k,\mathbf{y}^e(\xi_1,\xi_2)\right)}{\partial n_{\mathbf{x}}}N^a(\xi_1,\xi_2)J(\xi_1,\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2\cdot\mathbf{R}_a^e\end{aligned} \qquad (45)$$

where $A(e)$ is the number of nodes of the current element e (A = 8 or 6 for quadrilateral or triangular elements, respectively), N^a (a = 1, 2,…, A) are the shape functions of a typical quadrilateral or triangular quadratic element, J the corresponding Jacobian of the transformation from the global (X_1, X_2, X_3) to the local co-ordinate system (ξ_1, ξ_2) and $\mathbf{u}_a^e$, $\mathbf{q}_a^e$, $\mathbf{P}_a^e$ and $\mathbf{R}_a^e$ are the nodal values of the corresponding field functions. Adopting now a global numbering for the nodes, each pair (e, a) is associated to a number β and the integral equations (44) and (45) are written as

$$\frac{1}{2}\mathbf{u}^k + \sum_{\beta=1}^{L}\tilde{\mathbf{H}}_\beta^k \cdot \mathbf{u}^\beta + \sum_{\beta=1}^{L}\tilde{\mathbf{K}}_\beta^k \cdot \mathbf{q}^\beta = \sum_{\beta=1}^{L}\tilde{\mathbf{G}}_\beta^k \cdot \mathbf{P}^\beta + \sum_{\beta=1}^{L}\tilde{\mathbf{L}}_\beta^k \cdot \mathbf{R}^\beta \tag{46}$$

$$\frac{1}{2}\mathbf{q}^k + \sum_{\beta=1}^{L}\tilde{\mathbf{S}}_\beta^k \cdot \mathbf{u}^\beta + \sum_{\beta=1}^{L}\tilde{\mathbf{T}}_\beta^k \cdot \mathbf{q}^\beta = \sum_{\beta=1}^{L}\tilde{\mathbf{V}}_\beta^k \cdot \mathbf{P}^\beta + \sum_{\beta=1}^{L}\tilde{\mathbf{W}}_\beta^k \cdot \mathbf{R}^\beta \tag{47}$$

where L is the total number of nodes and

$$\tilde{\mathbf{H}}_\beta^k = \int_{-1}^{1}\int_{-1}^{1}\tilde{\tilde{\mathbf{P}}}^*\left(\mathbf{x}^k, \mathbf{y}^e(\xi_1,\xi_2)\right)N^a(\xi_1,\xi_2)J(\xi_1,\xi_2)\,\mathrm{d}\xi_1\mathrm{d}\xi_2\Bigg|_{(e,a)\to\beta} \tag{48}$$

$$\tilde{\mathbf{K}}_\beta^k = \int_{-1}^{1}\int_{-1}^{1}\tilde{\tilde{\mathbf{R}}}^*\left(\mathbf{x}^k, \mathbf{y}^e(\xi_1,\xi_2)\right)N^a(\xi_1,\xi_2)J(\xi_1,\xi_2)\,\mathrm{d}\xi_1\mathrm{d}\xi_2\Bigg|_{(e,a)\to\beta} \tag{49}$$

$$\tilde{\mathbf{G}}_\beta^k = \int_{-1}^{1}\int_{-1}^{1}\tilde{\mathbf{U}}^*\left(\mathbf{x}^k, \mathbf{y}^e(\xi_1,\xi_2)\right)N^a(\xi_1,\xi_2)J(\xi_1,\xi_2)\,\mathrm{d}\xi_1\mathrm{d}\xi_2\Bigg|_{(e,a)\to\beta} \tag{50}$$

$$\tilde{\mathbf{L}}_\beta^k = \int_{-1}^{1}\int_{-1}^{1}\tilde{\tilde{\mathbf{Q}}}^*\left(\mathbf{x}^k, \mathbf{y}^e(\xi_1,\xi_2)\right)N^a(\xi_1,\xi_2)J(\xi_1,\xi_2)\,\mathrm{d}\xi_1\mathrm{d}\xi_2\Bigg|_{(e,a)\to\beta} \tag{51}$$

$$\tilde{\mathbf{S}}_\beta^k = \int_{-1}^{1}\int_{-1}^{1}\frac{\partial\tilde{\tilde{\mathbf{P}}}^*\left(\mathbf{x}^k, \mathbf{y}^e(\xi_1,\xi_2)\right)}{\partial n_\mathbf{x}}N^a(\xi_1,\xi_2)J(\xi_1,\xi_2)\,\mathrm{d}\xi_1\mathrm{d}\xi_2\Bigg|_{(e,a)\to\beta} \tag{52}$$

$$\tilde{\mathbf{T}}_\beta^k = \int_{-1}^{1}\int_{-1}^{1}\frac{\partial\tilde{\tilde{\mathbf{R}}}^*\left(\mathbf{x}^k, \mathbf{y}^e(\xi_1,\xi_2)\right)}{\partial n_\mathbf{x}}N^a(\xi_1,\xi_2)J(\xi_1,\xi_2)\,\mathrm{d}\xi_1\mathrm{d}\xi_2\Bigg|_{(e,a)\to\beta} \tag{53}$$

$$\tilde{\mathbf{V}}_\beta^k = \int_{-1}^{1}\int_{-1}^{1}\frac{\partial\tilde{\mathbf{U}}^*\left(\mathbf{x}^k, \mathbf{y}^e(\xi_1,\xi_2)\right)}{\partial n_\mathbf{x}}N^a(\xi_1,\xi_2)J(\xi_1,\xi_2)\,\mathrm{d}\xi_1\mathrm{d}\xi_2\Bigg|_{(e,a)\to\beta} \tag{54}$$

$$\tilde{\mathbf{W}}_\beta^k = \int_{-1}^{1}\int_{-1}^{1}\frac{\partial\tilde{\tilde{\mathbf{Q}}}^*\left(\mathbf{x}^k, \mathbf{y}^e(\xi_1,\xi_2)\right)}{\partial n_\mathbf{x}}N^a(\xi_1,\xi_2)J(\xi_1,\xi_2)\,\mathrm{d}\xi_1\mathrm{d}\xi_2\Bigg|_{(e,a)\to\beta} \tag{55}$$

Collocating Eqs (46) and (47) at all nodal points L, one obtains the linear system of algebraic equations

$$\begin{bmatrix} \frac{1}{2}\tilde{\mathbf{I}} + \tilde{\mathbf{H}} & \tilde{\mathbf{K}} \\ \tilde{\mathbf{S}} & \frac{1}{2}\tilde{\mathbf{I}} + \tilde{\mathbf{T}} \end{bmatrix} \cdot \begin{Bmatrix} \mathbf{u} \\ \mathbf{q} \end{Bmatrix} = \begin{bmatrix} \tilde{\mathbf{G}} & \tilde{\mathbf{L}} \\ \tilde{\mathbf{V}} & \tilde{\mathbf{W}} \end{bmatrix} \cdot \begin{Bmatrix} \mathbf{P} \\ \mathbf{R} \end{Bmatrix} \tag{56}$$

where matrices $\widetilde{\mathbf{H}}$, $\widetilde{\mathbf{K}}$, $\widetilde{\mathbf{S}}$, $\widetilde{\mathbf{T}}$, $\widetilde{\mathbf{G}}$, $\widetilde{\mathbf{L}}$, $\widetilde{\mathbf{V}}$ and $\widetilde{\mathbf{W}}$ contain all the submatrices given by Eqs. (48)-(55), respectively. Applying the boundary conditions (8) and (9) and rearranging Eq. (56), one produces the final linear system of algebraic equations of the form

$$\widetilde{\mathbf{A}} \cdot \mathbf{X} = \mathbf{B} \tag{57}$$

where the vectors **X** and **B** contain all the unknown and known nodal components of the boundary fields,respectively and $\widetilde{\mathbf{A}}$ is a known influence matrix.

When $\beta \neq k$, integrals (48)-(55) are non-singular and can be easily computed numerically by Gauss quadrature, utilizing, as in the present work a 6×6 integration points scheme. In case $\beta = k$, the integrals (50), (51) and (54) are also non-singular, while the remaining integrals (48), (49), (52), (53) and (55) become singular with the first two being weakly singular integrals, the next two strongly singular (CPV) integrals and the last one a hypersingular integral. In the present work, the singular integrals are evaluated with high accuracy by applying a direct methodology explained in the work of Tsepoura et al. [14].

In order to demonstrate the accuracy of the proposed here 3-D gradient elastic boundary element methodology, a simple example dealing with the harmonic excitation of a solid sphere of radius a by a uniform external pressure P_0, is numerically solved. The classical boundary condition of the problem is

$$\mathbf{P}(\mathbf{y}) = P_0\, \hat{\mathbf{r}}, \qquad \mathbf{y} \in S_a \tag{58}$$

and the non-classical one

$$\mathbf{R}(\mathbf{y}) = \mathbf{0}, \qquad \mathbf{y} \in S_a \tag{59}$$

where S_a is the surface of the sphere. This problem can be easily solved analytically and its solution, as obtained by the present authors, has the form

$$u_r(r) = \frac{A}{a} j_1(k_1 r) + \frac{C}{a}\sqrt{\frac{\pi}{2(\ell_1 r)}}\, \mathrm{I}_{\frac{3}{2}}(\ell_1 r) \tag{60}$$

where

$$\begin{aligned}
&A = -2(1+\nu)aP_0 [k_1^2(-1+2\nu)a^3((1+g^2k_1^2)a(a^2+g^2(6+k_1^2a^2))\cosh(\ell_1 a) - \\
&3g\sqrt{1+g^2k_1^2}(a^2+g^2(2+k_1^2a^2))\sinh(\ell_1 a))] / [2\sqrt{1+g^2k_1^2}\,E(\sqrt{1+g^2k_1^2}\,a\cosh(\ell_1 a) \\
&(2k_1(1+2g^2k_1^2)a(3g^2(-3+4\nu)-(1-2\nu)a^2)\cos(k_1 a) + (-6g^2(1+2g^2k_1^2)(-3+4\nu)+ \\
&2(1-2\nu-g^2k_1^2(2+\nu)+g^4k_1^4(-6+7\nu))a^2 + (k_1+g^2k_1^3)^2(-1+\nu)a^4)\sin(k_1 a) + \\
&g(k_1 a(-6g^2(1+2g^2k_1^2)(-3+4\nu)-2(-3+6\nu+5g^2k_1^2(-2+3\nu)+g^4k_1^4(-6+7\nu))a^2 + \\
&g^2k_1^4(1+g^2k_1^2)(-1+\nu)a^4)\cos(k_1 a) - 3(1+2g^2k_1^2)(a^2(2-4\nu+k_1^2(-1+\nu)a^2)+ \\
&g^2(6-8\nu+k_1^4(-1+\nu)a^4))\sin(k_1 a)\sinh(\ell_1 a))]
\end{aligned} \tag{61}$$

$$
\begin{aligned}
&C = 2P_0 a(1+\nu)\left[g\left(1+g^2k_1^2\right)(-1+2\nu)a^3\left(k_1 a\left(-6+k_1^2a^2\right)\cos(k_1 a) - 3\left(-2+k_1^2a^2\right)\sin(k_1 a)\right)\right]/ \\
&\Big[2E\Big(\sqrt{1+g^2k_1^2}\,a\cosh(\ell_1 a)\Big(2k_1\left(1+2g^2k_1^2\right)a\left(3g^2(-3+4\nu)+(-1+2\nu)a^2\right)\cos(k_1 a) + \\
&\left(-6g^2\left(1+2g^2k_1^2\right)(-3+4\nu)+2\left(1-2\nu-g^2k_1^2(2+\nu)+g^4k_1^4(-6+7\nu)\right)a^2 + \right. \\
&\left.\left(k_1+g^2k_1^3\right)^2(-1+\nu)a^4\right)\sin(k_1 a)\Big) + g\Big(k_1 a\Big(-6g^2\left(1+2g^2k_1^2\right)(-3+4\nu) - \\
&2\left(-3+6\nu+5g^2k_1^2(-2+3\nu)+g^4k_1^4(-6+7\nu)\right)a^2 + g^2k_1^4\left(1+g^2k_1^2\right)(-1+\nu)a^4\Big)\cos(k_1 a) - \\
&3\left(1+2g^2k_1^2\right)\left(a^2\left(2-4\nu+k_1^2(-1+\nu)a^2\right)+g^2\left(6-8\nu+k_1^4(-1+\nu)a^4\right)\right)\sin(k_1 a)\Big)\sinh(\ell_1 a)\Big)\Big]
\end{aligned}
\tag{62}
$$

where $\ell_1 = \sqrt{1+g^2k_1^2}/g$, while $j_1(k_1 r)$ and $\sqrt{\pi/(2\ell_1 r)}\,I_{3/2}(\ell_1 r)$ are, the first order spherical Bessel function of the first kind and the first order modified Bessel function of the first kind, respectively.

Assuming $a=1$ and $P_0=1$ and discretizing only one octant of the sphere (due to the symmetry), the radial displacements as well as the radial strains for three values of the volumetric strain gradient energy coefficient g / α ($g/a = 0.05, 0.5, 1$) and for the excitation frequencies $\omega = 0.002 rad/\sec$ and $\omega = 40 rad/\sec$, are evaluated. Both displacement and strain radial fields are depicted in Figures 1, 3 and 2, 4, respectively, as functions of r/a and compared to the corresponding analytical ones. As it is evident from Figures 1-4 the agreement between the solutions is excellent. It is also observed that for low frequencies the gradient effect is negligible, while it becomes important for higher frequencies and small but non-zero values of the gradient coefficient.

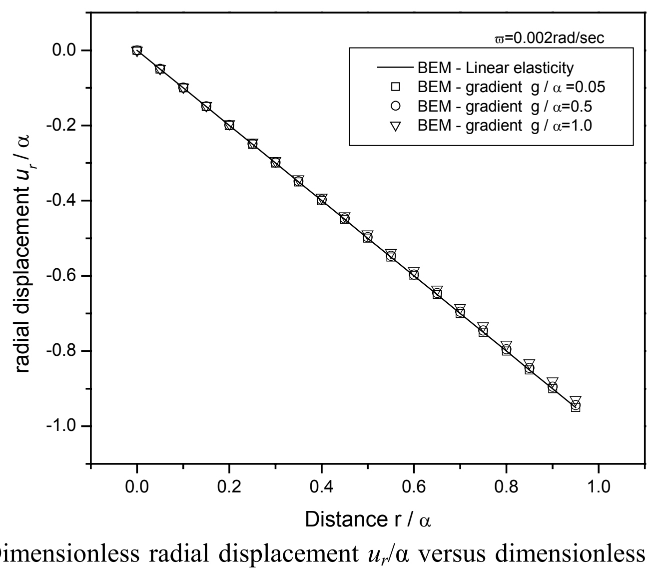

Figure 1: Dimensionless radial displacement u_r/α versus dimensionless radial distance r/α for various values of g/α and $\omega = 0.002 rad/\sec$.

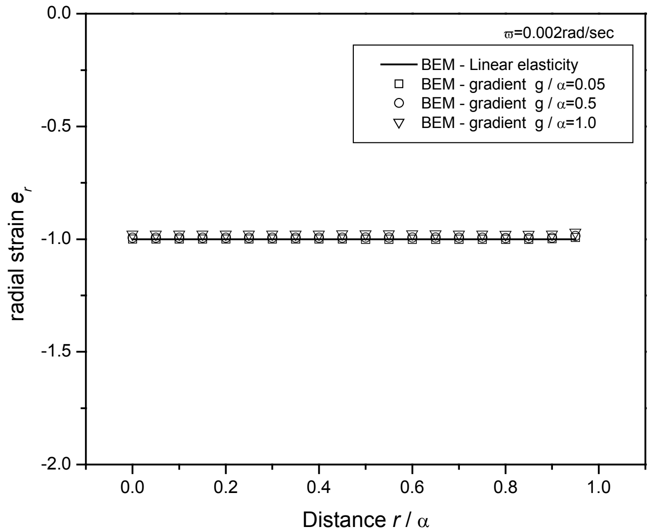

Figure 2: Dimensionless radial strain e_r versus dimensionless radial distance r/α for various values of g/α and $\omega = 0.002 rad/\sec$.

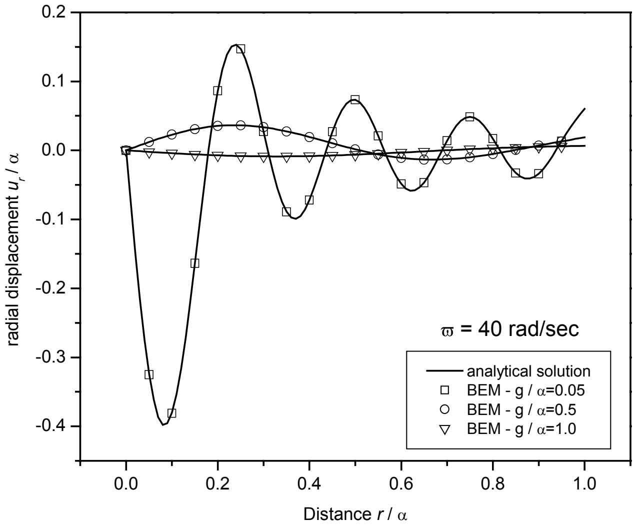

Figure 3: Dimensionless radial displacement u_r/α versus dimensionless radial distance r/α for various values of g/α and $\omega = 40 rad/\sec$.

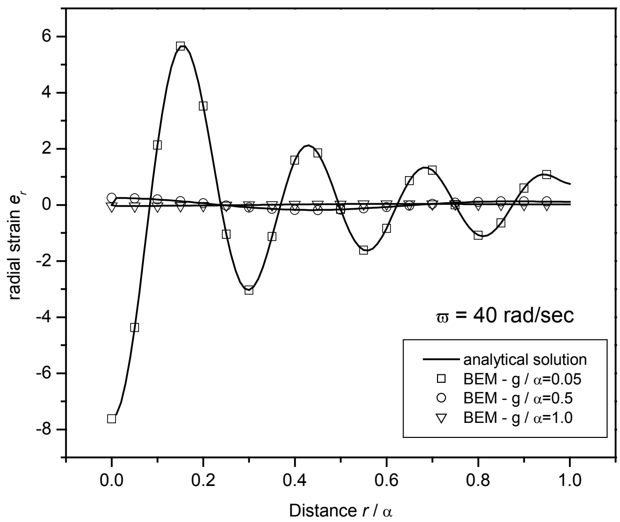

Figure 4: Dimensionless radial strain e_r versus dimensionless radial distance r/α for various values of g/α and $\omega = 40 rad/\sec$.

6. Conclusions

A boundary element method for solving three-dimensional static, gradient elastic problems in the frequency domain has been developed. Microstructural effects on the macroscopic behavior of the elastic material behavior have been taken into account by means of a simple gradient elastic theory due to Aifantis.

The equation of motion as well as all the possible boundary conditions (classical and non-classical) has been determined with the aid of a variational statement of the problem. The fundamental solution and the reciprocity identity of the gradient elastic problem have been explicitly determined. Both have been used to establish the boundary integral equation of the problem consisting of one equation for the displacement and another one for its normal derivative.

The numerical implementation of the problem is accomplished by discretizing the boundary of the problem into quadratic quadrilateral elements and employing advanced integration algorithms for the highly accurate evaluation of the singular integrals. A representative numerical example has been presented to illustrate the method and demonstrate its high accuracy.

Appendix I

In this Appendix the explicit expressions of the kernels appearing in the integral equation (41) are given as follows:

$$\widetilde{\mathbf{U}}^*(\mathbf{x},\mathbf{y}) = \frac{1}{16\pi\mu(1-\nu)}\left[\Psi\,\widetilde{\mathbf{I}} - \mathrm{X}\,\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\right] \tag{I.1}$$

$$\tilde{\tilde{\mathbf{Q}}}^*(\mathbf{x},\mathbf{y}) = \left(\frac{\partial \tilde{\mathbf{U}}^*(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}_y}\right)^T = \frac{1}{16\pi\mu(1-\nu)}\left[\left(\frac{2X}{r} - \frac{dX}{dr}\right)(\hat{\mathbf{n}}_y \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} \otimes \hat{\mathbf{r}} + \frac{d\Psi}{dr}(\hat{\mathbf{n}}_y \cdot \hat{\mathbf{r}})\tilde{\mathbf{I}} - \frac{X}{r}(\hat{\mathbf{n}}_y \otimes \hat{\mathbf{r}} + \hat{\mathbf{r}} \otimes \hat{\mathbf{n}}_y)\right] \tag{I.2}$$

$$\begin{aligned}\tilde{\tilde{\mathbf{R}}}^*(\mathbf{x},\mathbf{y}) = \frac{g^2}{16\pi(1-\nu)}\Bigg\{&\left[\left(\frac{dA}{dr} - \frac{3A}{r}\right)(\hat{\mathbf{n}}_y \cdot \hat{\mathbf{r}})^2 + \frac{A}{r}\right]\hat{\mathbf{r}} \otimes \hat{\mathbf{r}} + \left[\left(\frac{dB}{dr} - \frac{B}{r}\right)(\hat{\mathbf{n}}_y \cdot \hat{\mathbf{r}})^2 + \frac{B}{r}\right]\tilde{\mathbf{I}} + \\ &\left(\frac{dB}{dr} - \frac{B}{r} + \frac{A}{r}\right)(\hat{\mathbf{n}}_y \cdot \hat{\mathbf{r}})\hat{\mathbf{n}}_y \otimes \hat{\mathbf{r}} + \left(\frac{dC}{dr} - \frac{C}{r} + \frac{A}{r}\right)(\hat{\mathbf{n}}_y \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} \otimes \hat{\mathbf{n}}_y + \frac{B+C}{r}\hat{\mathbf{n}}_y \otimes \hat{\mathbf{n}}_y\Bigg\}\end{aligned} \tag{I.3}$$

$$\tilde{\tilde{\mathbf{P}}}^* = \left[\hat{\mathbf{n}}_y \cdot \tilde{\boldsymbol{\tau}}^* + (\hat{\mathbf{n}}_y \otimes \hat{\mathbf{n}}_y) : \frac{\partial \tilde{\tilde{\boldsymbol{\mu}}}^*}{\partial n_y} - \hat{\mathbf{n}}_y \cdot (\nabla \cdot \tilde{\tilde{\boldsymbol{\mu}}}^*) - \hat{\mathbf{n}}_y \cdot (\nabla \cdot \tilde{\tilde{\boldsymbol{\mu}}}^{*2134}) + (\nabla_S \cdot \hat{\mathbf{n}}_y)(\hat{\mathbf{n}}_y \otimes \hat{\mathbf{n}}_y) : \tilde{\tilde{\boldsymbol{\mu}}}^* - (\nabla_S \hat{\mathbf{n}}_y) : \tilde{\tilde{\boldsymbol{\mu}}}^*\right]^T \tag{I.4}$$

$$\begin{aligned}\tilde{\tilde{\mathbf{E}}}^*(\mathbf{x},\mathbf{y}) = \frac{g^2}{16\pi(1-\nu)}\Bigg\{&\left(\frac{dA}{dr} - \frac{3A}{r}\right)(\hat{\mathbf{n}} \cdot \hat{\mathbf{r}})(\hat{\mathbf{m}} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} \otimes \hat{\mathbf{r}} + \left(\frac{dB}{dr} - \frac{B}{r}\right)(\hat{\mathbf{n}} \cdot \hat{\mathbf{r}})(\hat{\mathbf{m}} \cdot \hat{\mathbf{r}})\tilde{\mathbf{I}} + \\ &\left(\frac{dB}{dr} - \frac{B}{r}\right)\hat{\mathbf{m}} \otimes \hat{\mathbf{r}} + \left(\frac{dC}{dr} - \frac{C}{r}\right)\hat{\mathbf{r}} \otimes \hat{\mathbf{m}} + \\ &\frac{A}{r}(\hat{\mathbf{m}} \cdot \hat{\mathbf{r}})(\hat{\mathbf{n}} \otimes \hat{\mathbf{r}} + \hat{\mathbf{r}} \otimes \hat{\mathbf{n}}) + \frac{C}{r}\hat{\mathbf{n}} \otimes \hat{\mathbf{m}} + \frac{B}{r}\hat{\mathbf{m}} \otimes \hat{\mathbf{n}}\Bigg\}\end{aligned} \tag{I.5}$$

The terms of kernel $\tilde{\mathbf{P}}^*$ are given as follows:

$$(\hat{\mathbf{n}}_y \cdot \tilde{\boldsymbol{\tau}}^*)^T = \frac{1}{16\pi(1-\nu)}\left[A(\hat{\mathbf{n}}_y \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} \otimes \hat{\mathbf{r}} + B(\hat{\mathbf{n}}_y \cdot \hat{\mathbf{r}})\tilde{\mathbf{I}} + B\,\hat{\mathbf{n}}_y \otimes \hat{\mathbf{r}} + C\,\hat{\mathbf{r}} \otimes \hat{\mathbf{n}}_y\right] \tag{I.6}$$

$$\left[\left(\hat{\mathbf{n}}_{\mathrm{y}} \otimes \hat{\mathbf{n}}_{\mathrm{y}}\right) : \frac{\partial \tilde{\tilde{\boldsymbol{\mu}}}^{*}}{\partial n_{\mathrm{y}}}\right]^{\mathrm{T}} = \frac{1}{16\pi(1-\nu)}\Big[\mathrm{G}_1 \hat{\mathbf{r}} \otimes \hat{\mathbf{r}} + \mathrm{G}_2 \tilde{\mathbf{I}} + \mathrm{G}_3 \hat{\mathbf{r}} \otimes \hat{\mathbf{n}} + \mathrm{G}_4 \hat{\mathbf{n}} \otimes \hat{\mathbf{r}} + \mathrm{G}_5 \hat{\mathbf{n}} \otimes \hat{\mathbf{n}}\Big] \tag{I.7}$$

$$\begin{aligned}\left[\hat{\mathbf{n}}_{\mathrm{y}} \cdot \left(\nabla \cdot \tilde{\tilde{\boldsymbol{\mu}}}^{*}\right)\right]^{\mathrm{T}} = \frac{g^2}{16\pi(1-\nu)}\Bigg[&\left(\frac{d^2A}{dr^2} + \frac{2}{r}\frac{dA}{dr} - \frac{12A}{r^2}\right)\left(\hat{\mathbf{n}}_{\mathrm{y}} \cdot \hat{\mathbf{r}}\right)\hat{\mathbf{r}} \otimes \hat{\mathbf{r}} + \\ &\left(\frac{d^2B}{dr^2} + \frac{2}{r}\frac{dB}{dr} - \frac{2B}{r^2} + \frac{2A}{r^2}\right)\left[\left(\hat{\mathbf{n}}_{\mathrm{y}} \cdot \hat{\mathbf{r}}\right)\tilde{\mathbf{I}} + \hat{\mathbf{n}}_{\mathrm{y}} \otimes \hat{\mathbf{r}}\right] + \\ &\left(\frac{d^2C}{dr^2} + \frac{2}{r}\frac{dC}{dr} - \frac{2C}{r^2} + \frac{2\mathrm{A}}{r^2}\right)\hat{\mathbf{r}} \otimes \hat{\mathbf{n}}_{\mathrm{y}}\Bigg]\end{aligned} \tag{I.8}$$

$$\left[\hat{\mathbf{n}}_{\mathrm{y}} \cdot \left(\nabla \cdot \tilde{\tilde{\boldsymbol{\mu}}}^{*2134}\right)\right]^{\mathrm{T}} = \frac{g^2}{16\pi(1-\nu)}\Big[D_1\left(\hat{\mathbf{n}}_{\mathrm{y}} \cdot \hat{\mathbf{r}}\right)\hat{\mathbf{r}} \otimes \hat{\mathbf{r}} + D_2\left(\hat{\mathbf{n}}_{\mathrm{y}} \cdot \hat{\mathbf{r}}\right)\tilde{\mathbf{I}} + D_3\left(\hat{\mathbf{n}}_{\mathrm{y}} \otimes \hat{\mathbf{r}} + \hat{\mathbf{r}} \otimes \hat{\mathbf{n}}_{\mathrm{y}}\right)\Big] \tag{I.9}$$

$$\begin{aligned}&\left[\left(\nabla_S \cdot \hat{\mathbf{n}}_{\mathrm{y}}\right)\left(\hat{\mathbf{n}}_{\mathrm{y}} \otimes \hat{\mathbf{n}}_{\mathrm{y}}\right) : \tilde{\tilde{\boldsymbol{\mu}}}^{*}\right]^{\mathrm{T}} = \\ &\quad \frac{g^2\left(\nabla_S \cdot \hat{\mathbf{n}}_{\mathrm{y}}\right)}{16\pi(1-\nu)}\Bigg\{\left[\left(\frac{dA}{dr} - \frac{3A}{r}\right)\left(\hat{\mathbf{n}}_{\mathrm{y}} \cdot \hat{\mathbf{r}}\right)^2 + \frac{A}{r}\right]\hat{\mathbf{r}} \otimes \hat{\mathbf{r}} + \\ &\quad \left[\left(\frac{dB}{dr} - \frac{B}{r}\right)\left(\hat{\mathbf{n}}_{\mathrm{y}} \cdot \hat{\mathbf{r}}\right)^2 + \frac{B}{r}\right]\tilde{\mathbf{I}} + \left(\frac{dB}{dr} - \frac{B}{r} + \frac{A}{r}\right)\left(\hat{\mathbf{n}}_{\mathrm{y}} \cdot \hat{\mathbf{r}}\right)\hat{\mathbf{n}}_{\mathrm{y}} \otimes \hat{\mathbf{r}} + \\ &\quad \left(\frac{dC}{dr} - \frac{C}{r} + \frac{A}{r}\right)\left(\hat{\mathbf{n}}_{\mathrm{y}} \cdot \hat{\mathbf{r}}\right)\hat{\mathbf{r}} \otimes \hat{\mathbf{n}}_{\mathrm{y}} + \frac{B+C}{r}\hat{\mathbf{n}}_{\mathrm{y}} \otimes \hat{\mathbf{n}}_{\mathrm{y}}\Bigg\}\end{aligned} \tag{I.10}$$

$$\begin{aligned}&\left[\left(\nabla_S \hat{\mathbf{n}}_{\mathrm{y}}\right) : \tilde{\tilde{\boldsymbol{\mu}}}^{*}\right]^{\mathrm{T}} = \\ &\quad \frac{g^2}{16\pi(1-\nu)}\Bigg\{\left[\left(\frac{dA}{dr} - \frac{3A}{r}\right)\left(\nabla_S \hat{\mathbf{n}}_{\mathrm{y}}\right) : \left(\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}\right) + \frac{A}{r}\left(\nabla_S \cdot \hat{\mathbf{n}}_{\mathrm{y}}\right)\right]\hat{\mathbf{r}} \otimes \hat{\mathbf{r}} + \\ &\quad \left[\left(\frac{dB}{dr} - \frac{B}{r}\right)\left(\nabla_S \hat{\mathbf{n}}_{\mathrm{y}}\right) : \left(\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}\right) + \frac{B}{r}\left(\nabla_S \cdot \hat{\mathbf{n}}_{\mathrm{y}}\right)\right]\tilde{\mathbf{I}} + \\ &\quad \frac{A}{r}\left[\left(\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}\right) \cdot \left(\nabla_S \hat{\mathbf{n}}_{\mathrm{y}}\right) + \hat{\mathbf{r}} \cdot \left(\nabla_S \hat{\mathbf{n}}_{\mathrm{y}}\right) \otimes \hat{\mathbf{r}}\right] + \left(\frac{dB}{dr} - \frac{B}{r}\right)\left(\nabla_S \hat{\mathbf{n}}_{\mathrm{y}}\right) \cdot \left(\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}\right) + \\ &\quad \left(\frac{dC}{dr} - \frac{C}{r}\right)\hat{\mathbf{r}} \otimes \left(\nabla_S \hat{\mathbf{n}}_{\mathrm{y}}\right) \cdot \hat{\mathbf{r}} + \frac{C}{r}\left(\nabla_S \hat{\mathbf{n}}_{\mathrm{y}}\right)^T + \frac{B}{r}\left(\nabla_S \hat{\mathbf{n}}_{\mathrm{y}}\right)\Bigg\}\end{aligned} \tag{I.11}$$

where

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r}, \quad \mathbf{r} = \mathbf{y} - \mathbf{x}, \quad r = |\mathbf{r}| \tag{I.12}$$

$$\mathrm{X} = -\frac{1}{r} + \frac{6g^2}{r^3} - \left(\frac{6g^2}{r^3} + \frac{6g}{r^2} + \frac{2}{r}\right)e^{-r/g} \tag{I.13}$$

$$\begin{aligned} \Psi = (3-4\nu)\frac{1}{r} + 2(1-2\nu)\left[-\frac{g^2}{r^3} + \left(\frac{g^2}{r^3} + \frac{g}{r^2}\right)e^{-r/g}\right] + \\ 4(1-\nu)\left[\frac{g^2}{r^3} - \left(\frac{g^2}{r^3} + \frac{g}{r^2} + \frac{1}{r}\right)e^{-r/g}\right] \end{aligned} \tag{I.14}$$

$$\begin{aligned} A = 2\left(\frac{2\mathrm{X}}{r} - \frac{d\mathrm{X}}{dr}\right), \quad B = \frac{d\Psi}{dr} - \frac{\mathrm{X}}{r}, \\ C = \frac{2\nu}{1-2\nu}\left(\frac{d\Psi}{dr} - \frac{d\mathrm{X}}{dr} - \frac{2\mathrm{X}}{r}\right) - \frac{2\mathrm{X}}{r} \end{aligned} \tag{I.15}$$

$$\mathrm{G}_1 = g^2\left\{\left(\frac{d^2A}{dr^2} - \frac{7}{r}\frac{dA}{dr} + \frac{15A}{r^2}\right)(\hat{\mathbf{n}}_y \cdot \hat{\mathbf{r}})^3 + 3\left(\frac{1}{r}\frac{dA}{dr} - \frac{3A}{r^2}\right)(\hat{\mathbf{n}}_y \cdot \hat{\mathbf{r}})\right\} \tag{I.16}$$

$$\mathrm{G}_2 = g^2\left\{\left(\frac{d^2B}{dr^2} - \frac{3}{r}\frac{dB}{dr} + \frac{3B}{r^2}\right)(\hat{\mathbf{n}}_y \cdot \hat{\mathbf{r}})^3 + 3\left(\frac{1}{r}\frac{dB}{dr} - \frac{B}{r^2}\right)(\hat{\mathbf{n}}_y \cdot \hat{\mathbf{r}})\right\} \tag{I.17}$$

$$\begin{aligned} \mathrm{G}_3 = g^2\left\{\left(\frac{d^2C}{dr^2} - \frac{3}{r}\frac{dC}{dr} + \frac{3C}{r^2} + \frac{2}{r}\frac{dA}{dr} - \frac{6A}{r^2}\right)(\hat{\mathbf{n}}_y \cdot \hat{\mathbf{r}})^2 + \right. \\ \left. \frac{2A}{r^2} + \frac{1}{r}\frac{dC}{dr} - \frac{C}{r^2}\right\} \end{aligned} \tag{I.18}$$

$$\begin{aligned} \mathrm{G}_4 = g^2\left\{\left(\frac{d^2B}{dr^2} - \frac{3}{r}\frac{dB}{dr} + \frac{3B}{r^2} + \frac{2}{r}\frac{dA}{dr} - \frac{6A}{r^2}\right)(\hat{\mathbf{n}}_y \cdot \hat{\mathbf{r}})^2 + \right. \\ \left. \frac{2A}{r^2} + \frac{1}{r}\frac{dB}{dr} - \frac{B}{r^2}\right\} \end{aligned} \tag{I.19}$$

$$\mathrm{G}_5 = g^2\left\{2\left(\frac{1}{r}\frac{dB}{dr} - \frac{B}{r^2} + \frac{1}{r}\frac{dC}{dr} - \frac{C}{r^2} + \frac{A}{r^2}\right)(\hat{\mathbf{n}}_y \cdot \hat{\mathbf{r}})\right\} \tag{I.20}$$

$$D_1 = \frac{\mathrm{d}^2 A}{\mathrm{d}r^2} - \frac{6A}{r^2} + \frac{\mathrm{d}^2 B}{\mathrm{d}r^2} - \frac{3}{r}\frac{\mathrm{d}B}{\mathrm{d}r} + \frac{3B}{r^2} + \frac{\mathrm{d}^2 C}{\mathrm{d}r^2} - \frac{3}{r}\frac{\mathrm{d}C}{\mathrm{d}r} + \frac{3C}{r^2}$$

$$D_2 = \frac{\mathrm{d}^2 B}{\mathrm{d}r^2} + \frac{3}{r}\frac{\mathrm{d}B}{\mathrm{d}r} - \frac{3B}{r^2} + \frac{1}{r}\frac{\mathrm{d}C}{\mathrm{d}r} - \frac{C}{r^2} \qquad \text{(I.21)}$$

$$D_3 = \frac{1}{r}\frac{\mathrm{d}A}{\mathrm{d}r} + \frac{2A}{r^2} + \frac{1}{r}\frac{\mathrm{d}B}{\mathrm{d}r} - \frac{B}{r^2} + \frac{1}{r}\frac{\mathrm{d}C}{\mathrm{d}r} - \frac{C}{r^2}$$

$$D_4 = \frac{\mathrm{d}A}{\mathrm{d}r} + \frac{2A}{r} + \frac{\mathrm{d}B}{\mathrm{d}r} - \frac{B}{r} + \frac{\mathrm{d}C}{\mathrm{d}r} - \frac{C}{r}$$

$$D_5 = \frac{\mathrm{d}B}{\mathrm{d}r} + \frac{3B}{r} + \frac{C}{r}$$

APPENDIX II

In this Appendix the explicit expressions of the kernels appearing in the integral equation (43) are given as follows:

$$\frac{\partial\left[\tilde{\mathbf{U}}^*(\mathbf{x},\mathbf{y})\right]}{\partial n_{\mathbf{x}}} = \frac{1}{16\pi\mu(1-\nu)}\left[-\frac{d\Psi}{dr}(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})\tilde{\mathbf{I}} + \left(\frac{d\mathrm{X}}{dr} - \frac{2}{r}\mathrm{X}\right)(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\otimes\hat{\mathbf{r}} + \frac{\mathrm{X}}{r}(\hat{\mathbf{n}}_{\mathbf{x}}\otimes\hat{\mathbf{r}} + \hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_{\mathbf{x}})\right] \qquad \text{(II.1)}$$

$$\begin{aligned}\frac{\partial\left[\tilde{\tilde{\mathbf{Q}}}^*(\mathbf{x},\mathbf{y})\right]}{\partial n_{\mathbf{x}}} = \frac{1}{16\pi\mu(1-\nu)}\Bigg[&-\left(\frac{5}{r}\frac{d\mathrm{X}}{dr} - \frac{8}{r^2}\mathrm{X} - \frac{d^2\mathrm{X}}{dr^2}\right)(\hat{\mathbf{n}}_{\mathbf{y}}\cdot\hat{\mathbf{r}})(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\otimes\hat{\mathbf{r}} - \\ &\left(\frac{2}{r^2}\mathrm{X} - \frac{1}{r}\frac{d\mathrm{X}}{dr}\right)(\hat{\mathbf{n}}_{\mathbf{y}}\cdot\hat{\mathbf{n}}_{\mathbf{x}})\hat{\mathbf{r}}\otimes\hat{\mathbf{r}} - \left(\frac{2}{r^2}\mathrm{X} - \frac{1}{r}\frac{d\mathrm{X}}{dr}\right)(\hat{\mathbf{n}}_{\mathbf{y}}\cdot\hat{\mathbf{r}})(\hat{\mathbf{n}}_{\mathbf{x}}\otimes\hat{\mathbf{r}} + \hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_{\mathbf{x}}) - \\ &\left(\frac{d^2\Psi}{dr^2} - \frac{1}{r}\frac{d\Psi}{dr}\right)(\hat{\mathbf{n}}_{\mathbf{y}}\cdot\hat{\mathbf{r}})(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})\tilde{\mathbf{I}} - \frac{1}{r}\frac{d\Psi}{dr}(\hat{\mathbf{n}}_{\mathbf{y}}\cdot\hat{\mathbf{n}}_{\mathbf{x}})\tilde{\mathbf{I}} - \\ &\left(\frac{2}{r^2}\mathrm{X} - \frac{1}{r}\frac{d\mathrm{X}}{dr}\right)(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})\hat{\mathbf{n}}_{\mathbf{y}}\otimes\hat{\mathbf{r}} + \frac{1}{r^2}\mathrm{X}\hat{\mathbf{n}}_{\mathbf{y}}\otimes\hat{\mathbf{n}}_{\mathbf{x}} - \\ &\left(\frac{2}{r^2}\mathrm{X} - \frac{1}{r}\frac{d\mathrm{X}}{dr}\right)(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_{\mathbf{y}} + \frac{1}{r^2}\mathrm{X}\hat{\mathbf{n}}_{\mathbf{x}}\otimes\hat{\mathbf{n}}_{\mathbf{y}}\Bigg]\end{aligned} \qquad \text{(II.2)}$$

$$\frac{\partial\left[\tilde{\tilde{\mathbf{R}}}^*(\mathbf{x},\mathbf{y})\right]}{\partial n_\mathbf{x}}=\frac{g^2}{16\pi(1-\nu)}\left[-\left(\frac{dA_1}{dr}-\frac{4A_1}{r}\right)(\hat{\mathbf{n}}_\mathrm{y}\cdot\hat{\mathbf{r}})^2(\hat{\mathbf{n}}_\mathrm{x}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}-\right.$$
$$\frac{2A_1}{r}(\hat{\mathbf{n}}_\mathrm{y}\cdot\hat{\mathbf{r}})(\hat{\mathbf{n}}_\mathrm{y}\cdot\hat{\mathbf{n}}_\mathrm{x})\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}-$$
$$\frac{A_1}{r}(\hat{\mathbf{n}}_\mathrm{y}\cdot\hat{\mathbf{r}})^2(\hat{\mathbf{n}}_\mathrm{x}\otimes\hat{\mathbf{r}}+\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_\mathrm{x})-\left(\frac{dA_6}{dr}-\frac{2A_6}{r}\right)(\hat{\mathbf{n}}_\mathrm{x}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}-$$
$$\frac{A_6}{r}(\hat{\mathbf{n}}_\mathrm{x}\otimes\hat{\mathbf{r}}+\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_\mathrm{x})-\left(\frac{dA_2}{dr}-\frac{2A_2}{r}\right)(\hat{\mathbf{n}}_\mathrm{y}\cdot\hat{\mathbf{r}})^2(\hat{\mathbf{n}}_\mathrm{x}\cdot\hat{\mathbf{r}})\tilde{\mathbf{I}}-$$
$$\frac{2A_2}{r}(\hat{\mathbf{n}}_\mathrm{y}\cdot\hat{\mathbf{r}})(\hat{\mathbf{n}}_\mathrm{y}\cdot\hat{\mathbf{n}}_\mathrm{x})\tilde{\mathbf{I}}-\frac{dA_7}{dr}(\hat{\mathbf{n}}_\mathrm{x}\cdot\hat{\mathbf{r}})\tilde{\mathbf{I}}-$$
$$\left(\frac{dA_3}{dr}-\frac{2A_3}{r}\right)(\hat{\mathbf{n}}_\mathrm{y}\cdot\hat{\mathbf{r}})(\hat{\mathbf{n}}_\mathrm{x}\cdot\hat{\mathbf{r}})\hat{\mathbf{n}}_\mathrm{y}\otimes\hat{\mathbf{r}}-\frac{A_3}{r}(\hat{\mathbf{n}}_\mathrm{y}\cdot\hat{\mathbf{n}}_\mathrm{x})\hat{\mathbf{n}}_\mathrm{y}\otimes\hat{\mathbf{r}}-$$
$$\frac{A_3}{r}(\hat{\mathbf{n}}_\mathrm{y}\cdot\hat{\mathbf{r}})\hat{\mathbf{n}}_\mathrm{y}\otimes\hat{\mathbf{n}}_\mathrm{x}-\left(\frac{dA_4}{dr}-\frac{2A_4}{r}\right)(\hat{\mathbf{n}}_\mathrm{y}\cdot\hat{\mathbf{r}})(\hat{\mathbf{n}}_\mathrm{x}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_\mathrm{y}-$$
$$\left.\frac{A_4}{r}(\hat{\mathbf{n}}_\mathrm{y}\cdot\hat{\mathbf{n}}_\mathrm{x})\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_\mathrm{y}-\frac{A_4}{r}(\hat{\mathbf{n}}_\mathrm{y}\cdot\hat{\mathbf{r}})\hat{\mathbf{n}}_\mathrm{x}\otimes\hat{\mathbf{n}}_\mathrm{y}-\frac{dA_5}{dr}(\hat{\mathbf{n}}_\mathrm{x}\cdot\hat{\mathbf{r}})\hat{\mathbf{n}}_\mathrm{y}\otimes\hat{\mathbf{n}}_\mathrm{y}\right] \tag{II.3}$$

$$\frac{\partial\left[\tilde{\tilde{\mathbf{P}}}^*(\mathbf{x},\mathbf{y})\right]}{\partial n_\mathbf{x}}=\frac{\partial}{\partial n_\mathbf{x}}\left[\hat{\mathbf{n}}_\mathrm{y}\cdot\tilde{\boldsymbol{\tau}}^*+(\hat{\mathbf{n}}_\mathrm{y}\otimes\hat{\mathbf{n}}_\mathrm{y}):\frac{\partial\tilde{\tilde{\boldsymbol{\mu}}}^*}{\partial n_\mathrm{y}}-\hat{\mathbf{n}}_\mathrm{y}\cdot\left(\nabla\cdot\tilde{\tilde{\boldsymbol{\mu}}}^*\right)\right.$$
$$\left.-\hat{\mathbf{n}}_\mathrm{y}\cdot\left(\nabla\cdot\tilde{\tilde{\boldsymbol{\mu}}}^{*2134}\right)+\left(\nabla_S\cdot\hat{\mathbf{n}}_\mathrm{y}\right)(\hat{\mathbf{n}}_\mathrm{y}\otimes\hat{\mathbf{n}}_\mathrm{y}):\tilde{\tilde{\boldsymbol{\mu}}}^*-\left(\nabla_S\hat{\mathbf{n}}_\mathrm{y}\right):\tilde{\tilde{\boldsymbol{\mu}}}^*\right]^T \tag{II.4}$$

$$\frac{\partial\left[\tilde{\tilde{\mathbf{E}}}^*(\mathbf{x},\mathbf{y})\right]}{\partial n_\mathbf{x}}=\frac{g^2}{16\pi(1-\nu)}\left\{-\left[\left(\frac{dA_1}{dr}-\frac{4\mathrm{A}_1}{r}\right)(\hat{\mathbf{n}}_\mathrm{x}\cdot\hat{\mathbf{r}})(\hat{\mathbf{n}}_\mathrm{y}\cdot\hat{\mathbf{r}})(\hat{\mathbf{m}}_\mathrm{y}\cdot\hat{\mathbf{r}})+\right.\right.$$
$$\left.\frac{\mathrm{A}_1}{r}(\hat{\mathbf{n}}_\mathrm{x}\cdot\hat{\mathbf{n}}_\mathrm{y})(\hat{\mathbf{m}}_\mathrm{y}\cdot\hat{\mathbf{r}})+\frac{\mathrm{A}_1}{r}(\hat{\mathbf{n}}_\mathrm{x}\cdot\hat{\mathbf{m}}_\mathrm{y})(\hat{\mathbf{n}}_\mathrm{y}\cdot\hat{\mathbf{r}})\right]\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}-$$
$$\frac{\mathrm{A}_1}{r}(\hat{\mathbf{n}}_\mathrm{y}\cdot\hat{\mathbf{r}})(\hat{\mathbf{m}}_\mathrm{y}\cdot\hat{\mathbf{r}})(\hat{\mathbf{n}}_\mathrm{x}\otimes\hat{\mathbf{r}}+\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_\mathrm{x})-$$
$$\left[\left(\frac{dA_2}{dr}-\frac{2\mathrm{A}_2}{r}\right)(\hat{\mathbf{n}}_\mathrm{x}\cdot\hat{\mathbf{r}})(\hat{\mathbf{n}}_\mathrm{y}\cdot\hat{\mathbf{r}})(\hat{\mathbf{m}}_\mathrm{y}\cdot\hat{\mathbf{r}})+\frac{\mathrm{A}_2}{r}(\hat{\mathbf{n}}_\mathrm{x}\cdot\hat{\mathbf{n}}_\mathrm{y})(\hat{\mathbf{m}}_\mathrm{y}\cdot\hat{\mathbf{r}})\right.$$
$$\left.+\frac{\mathrm{A}_1}{r}(\hat{\mathbf{n}}_\mathrm{x}\cdot\hat{\mathbf{m}}_\mathrm{y})(\hat{\mathbf{n}}_\mathrm{y}\cdot\hat{\mathbf{r}})\right]\tilde{\mathbf{I}}-\left(\frac{dA_2}{dr}-\frac{\mathrm{A}_2}{r}\right)(\hat{\mathbf{n}}_\mathrm{x}\cdot\hat{\mathbf{r}})\hat{\mathbf{m}}_\mathrm{y}\otimes\hat{\mathbf{r}}-$$

$$\begin{aligned}
&\frac{A_2}{r}\hat{\mathbf{m}}_y \otimes \hat{\mathbf{n}}_x - \left(\frac{dA_9}{dr} - \frac{A_9}{r}\right)(\hat{\mathbf{n}}_x \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} \otimes \hat{\mathbf{m}}_y - \frac{A_9}{r}\hat{\mathbf{n}}_x \otimes \hat{\mathbf{m}}_y - \\
&\left[\left(\frac{dA_6}{dr} - \frac{2A_6}{r}\right)(\hat{\mathbf{n}}_x \cdot \hat{\mathbf{r}})(\hat{\mathbf{m}}_y \cdot \hat{\mathbf{r}}) + \frac{A_6}{r}(\hat{\mathbf{m}}_y \cdot \hat{\mathbf{n}}_x)\right]\hat{\mathbf{n}}_y \otimes \hat{\mathbf{r}} - \\
&\frac{A_6}{r}(\hat{\mathbf{m}}_y \cdot \hat{\mathbf{r}})\hat{\mathbf{n}}_y \otimes \hat{\mathbf{n}}_x - \\
&\left[\left(\frac{dA_6}{dr} - \frac{2A_6}{r}\right)(\hat{\mathbf{n}}_x \cdot \hat{\mathbf{r}})(\hat{\mathbf{m}}_y \cdot \hat{\mathbf{r}}) + \frac{A_6}{r}(\hat{\mathbf{m}}_y \cdot \hat{\mathbf{n}}_x)\right]\hat{\mathbf{r}} \otimes \hat{\mathbf{n}}_y - \\
&\left.\frac{A_6}{r}(\hat{\mathbf{m}}_y \cdot \hat{\mathbf{r}})\hat{\mathbf{n}}_x \otimes \hat{\mathbf{n}}_y - \frac{dA_8}{dr}(\hat{\mathbf{n}}_x \cdot \hat{\mathbf{r}})\hat{\mathbf{n}}_y \otimes \hat{\mathbf{m}}_y - \frac{dA_7}{dr}(\hat{\mathbf{n}}_x \cdot \hat{\mathbf{r}})\hat{\mathbf{m}}_y \otimes \hat{\mathbf{n}}_y \right\}
\end{aligned} \tag{II.5}$$

The terms of kernel $\dfrac{\partial\left[\tilde{\tilde{\mathbf{P}}}^*(\mathbf{x},\mathbf{y})\right]}{\partial n_{\mathbf{x}}}$ are given as follows:

$$\begin{aligned}
\frac{\partial(\hat{\mathbf{n}}_y \cdot \tilde{\boldsymbol{\tau}}^*)^T}{\partial n_x} = \frac{g^2}{16\pi(1-\nu)} &\left[-\left(\frac{dA}{dr} - \frac{3A}{r}\right)(\hat{\mathbf{n}}_y \cdot \hat{\mathbf{r}})(\hat{\mathbf{n}}_x \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} \otimes \hat{\mathbf{r}} - \right.\\
&\frac{A}{r}(\hat{\mathbf{n}}_y \cdot \hat{\mathbf{n}}_x)\hat{\mathbf{r}} \otimes \hat{\mathbf{r}} - \frac{A}{r}(\hat{\mathbf{n}}_y \cdot \hat{\mathbf{r}})(\hat{\mathbf{n}}_x \otimes \hat{\mathbf{r}} + \hat{\mathbf{r}} \otimes \hat{\mathbf{n}}_x) - \\
&\left(\frac{dB}{dr} - \frac{B}{r}\right)(\hat{\mathbf{n}}_y \cdot \hat{\mathbf{r}})(\hat{\mathbf{n}}_x \cdot \hat{\mathbf{r}})\tilde{\mathbf{I}} - \frac{B}{r}(\hat{\mathbf{n}}_y \cdot \hat{\mathbf{n}}_x)\tilde{\mathbf{I}} - \\
&\left(\frac{dB}{dr} - \frac{B}{r}\right)(\hat{\mathbf{n}}_x \cdot \hat{\mathbf{r}})\hat{\mathbf{n}}_y \otimes \hat{\mathbf{r}} - \frac{B}{r}\hat{\mathbf{n}}_y \otimes \hat{\mathbf{n}}_x - \\
&\left.\left(\frac{dC}{dr} - \frac{C}{r}\right)(\hat{\mathbf{n}}_x \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} \otimes \hat{\mathbf{n}}_y - \frac{C}{r}\hat{\mathbf{n}}_x \otimes \hat{\mathbf{n}}_y\right]
\end{aligned} \tag{II.6}$$

$$
\begin{aligned}
&\frac{\partial}{\partial n_{\mathbf{x}}}\left[\left(\hat{\mathbf{n}}_{\mathrm{y}}\otimes\hat{\mathbf{n}}_{\mathrm{y}}\right):\frac{\partial\widetilde{\widetilde{\boldsymbol{\mu}}}^{*}}{\partial n_{\mathrm{y}}}\right]^{T}=\frac{g^{2}}{16\pi(1-\nu)}\left[-\left(\frac{dB_1}{dr}-\frac{5B_1}{r}\right)\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}}\right)^{3}\left(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}-\right.\\
&\frac{3B_1}{r}\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}}\right)^{2}\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{n}}_{\mathbf{x}}\right)\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}-\frac{B_1}{r}\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}}\right)^{3}\left(\hat{\mathbf{n}}_{\mathbf{x}}\otimes\hat{\mathbf{r}}+\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_{\mathbf{x}}\right)-\\
&\left(\frac{dB_2}{dr}-\frac{3B_2}{r}\right)\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}}\right)\left(\hat{\mathbf{n}}_{\mathrm{x}}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}-\frac{B_2}{r}\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{n}}_{\mathbf{x}}\right)\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}-\\
&\frac{B_2}{r}\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}}\right)\left(\hat{\mathbf{n}}_{\mathbf{x}}\otimes\hat{\mathbf{r}}+\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_{\mathbf{x}}\right)-\left(\frac{dB_3}{dr}-\frac{3B_3}{r}\right)\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}}\right)^{3}\left(\hat{\mathbf{n}}_{\mathrm{x}}\cdot\hat{\mathbf{r}}\right)\widetilde{\mathbf{I}}-\\
&\frac{3B_3}{r}\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}}\right)^{2}\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{n}}_{\mathbf{x}}\right)\widetilde{\mathbf{I}}-\left(\frac{dB_4}{dr}-\frac{B_4}{r}\right)\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}}\right)\left(\hat{\mathbf{n}}_{\mathrm{x}}\cdot\hat{\mathbf{r}}\right)\widetilde{\mathbf{I}}-\frac{B_4}{r}\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{n}}_{\mathrm{x}}\right)\widetilde{\mathbf{I}}-\\
&\left(\frac{dB_5}{dr}-\frac{3B_5}{r}\right)\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}}\right)^{2}\left(\hat{\mathbf{n}}_{\mathrm{x}}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_{\mathrm{y}}-\frac{2B_5}{r}\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}}\right)\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{n}}_{\mathbf{x}}\right)\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_{\mathrm{y}}-\\
&\frac{B_5}{r}\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}}\right)^{2}\hat{\mathbf{n}}_{\mathbf{x}}\otimes\hat{\mathbf{n}}_{\mathrm{y}}-\left(\frac{dB_6}{dr}-\frac{B_6}{r}\right)\left(\hat{\mathbf{n}}_{\mathrm{x}}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_{\mathrm{y}}-\frac{B_6}{r}\hat{\mathbf{n}}_{\mathbf{x}}\otimes\hat{\mathbf{n}}_{\mathrm{y}}-\\
&\left(\frac{dB_7}{dr}-\frac{3B_7}{r}\right)\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}}\right)^{2}\left(\hat{\mathbf{n}}_{\mathrm{x}}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{n}}_{\mathrm{y}}\otimes\hat{\mathbf{r}}-\frac{2B_7}{r}\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}}\right)\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{n}}_{\mathbf{x}}\right)\hat{\mathbf{n}}_{\mathrm{y}}\otimes\hat{\mathbf{r}}-\\
&\frac{B_7}{r}\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}}\right)^{2}\hat{\mathbf{n}}_{\mathrm{y}}\otimes\hat{\mathbf{n}}_{\mathbf{x}}-\left(\frac{dB_8}{dr}-\frac{B_8}{r}\right)\left(\hat{\mathbf{n}}_{\mathrm{x}}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{n}}_{\mathrm{y}}\otimes\hat{\mathbf{r}}-\frac{B_8}{r}\hat{\mathbf{n}}_{\mathrm{y}}\otimes\hat{\mathbf{n}}_{\mathbf{x}}-\\
&\left.\left(\frac{dB_9}{dr}-\frac{B_9}{r}\right)\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}}\right)\left(\hat{\mathbf{n}}_{\mathrm{x}}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{n}}_{\mathrm{y}}\otimes\hat{\mathbf{n}}_{\mathrm{y}}-\frac{B_9}{r}\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{n}}_{\mathrm{x}}\right)\hat{\mathbf{n}}_{\mathrm{y}}\otimes\hat{\mathbf{n}}_{\mathrm{y}}\right]
\end{aligned}
\qquad \text{(II.7)}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial n_{\mathbf{x}}}\left[\hat{\mathbf{n}}_{\mathrm{y}}\cdot\left(\nabla\cdot\widetilde{\widetilde{\boldsymbol{\mu}}}^{*}\right)\right]^{T}=\frac{g^{2}}{16\pi(1-\nu)}\\
&\left[-\left(\frac{dF_1}{dr}-\frac{3F_1}{r}\right)\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}}\right)\left(\hat{\mathbf{n}}_{\mathrm{x}}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}-\frac{F_1}{r}\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{n}}_{\mathbf{x}}\right)\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}-\right.\\
&\frac{F_1}{r}\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}}\right)\left(\hat{\mathbf{n}}_{\mathbf{x}}\otimes\hat{\mathbf{r}}+\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_{\mathbf{x}}\right)-\left(\frac{dF_2}{dr}-\frac{F_2}{r}\right)\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}}\right)\left(\hat{\mathbf{n}}_{\mathrm{x}}\cdot\hat{\mathbf{r}}\right)\widetilde{\mathbf{I}}-\\
&\frac{F_2}{r}\left(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{n}}_{\mathrm{x}}\right)\widetilde{\mathbf{I}}-\left(\frac{dF_2}{dr}-\frac{F_2}{r}\right)\left(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{n}}_{\mathrm{y}}\otimes\hat{\mathbf{r}}-\\
&\left.\frac{F_2}{r}\hat{\mathbf{n}}_{\mathrm{y}}\otimes\hat{\mathbf{n}}_{\mathbf{x}}-\left(\frac{dF_3}{dr}-\frac{F_3}{r}\right)\left(\hat{\mathbf{n}}_{\mathrm{x}}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_{\mathrm{y}}-\frac{F_3}{r}\hat{\mathbf{n}}_{\mathbf{x}}\otimes\hat{\mathbf{n}}_{\mathrm{y}}\right]
\end{aligned}
\qquad \text{(II.8)}
$$

$$\frac{\partial}{\partial n_{\mathbf{x}}}\left[\hat{\mathbf{n}}_{\mathrm{y}}\cdot\left(\nabla\cdot\tilde{\tilde{\boldsymbol{\mu}}}^{*2134}\right)\right]^{T}=\frac{g^{2}}{16\pi(1-\nu)}\left[-\left(\frac{dD_{1}}{dr}-\frac{3D_{1}}{r}\right)(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}})(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}-\right.$$
$$\frac{D_{1}}{r}(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{n}}_{\mathbf{x}})\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}-\frac{D_{1}}{r}(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}})(\hat{\mathbf{n}}_{\mathbf{x}}\otimes\hat{\mathbf{r}}+\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_{\mathbf{x}})- \quad \text{(II.9)}$$
$$\left(\frac{dD_{2}}{dr}-\frac{D_{2}}{r}\right)(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}})(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})\tilde{\mathbf{I}}-\frac{D_{2}}{r}(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{n}}_{\mathbf{x}})\tilde{\mathbf{I}}-$$
$$\left.\left(\frac{dD_{3}}{dr}-\frac{D_{3}}{r}\right)(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})(\hat{\mathbf{n}}_{\mathrm{y}}\otimes\hat{\mathbf{r}}+\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_{\mathrm{y}})-\frac{D_{3}}{r}(\hat{\mathbf{n}}_{\mathrm{y}}\otimes\hat{\mathbf{n}}_{\mathbf{x}}+\hat{\mathbf{n}}_{\mathbf{x}}\otimes\hat{\mathbf{n}}_{\mathrm{y}})\right]$$

$$\frac{\partial}{\partial n_{\mathbf{x}}}\left[(\nabla_{S}\cdot\hat{\mathbf{n}}_{\mathrm{y}})(\hat{\mathbf{n}}_{\mathrm{y}}\otimes\hat{\mathbf{n}}_{\mathrm{y}}):\tilde{\tilde{\boldsymbol{\mu}}}^{*}\right]^{T}=\frac{g^{2}(\nabla_{S}\cdot\hat{\mathbf{n}}_{\mathrm{y}})}{16\pi(1-\nu)}\left[-\left(\frac{dA_{1}}{dr}-\frac{4A_{1}}{r}\right)(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}})^{2}(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}-\right.$$
$$\frac{2A_{1}}{r}(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}})(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{n}}_{\mathbf{x}})\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}-\frac{A_{1}}{r}(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}})^{2}(\hat{\mathbf{n}}_{\mathbf{x}}\otimes\hat{\mathbf{r}}+\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_{\mathbf{x}})-$$
$$\left(\frac{dA_{6}}{dr}-\frac{2A_{6}}{r}\right)(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}-\frac{A_{6}}{r}(\hat{\mathbf{n}}_{\mathbf{x}}\otimes\hat{\mathbf{r}}+\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_{\mathbf{x}})-$$
$$\left(\frac{dA_{2}}{dr}-\frac{2A_{2}}{r}\right)(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}})^{2}(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})\tilde{\mathbf{I}}-\frac{2A_{2}}{r}(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}})(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{n}}_{\mathbf{x}})\tilde{\mathbf{I}}-$$
$$\frac{dA_{7}}{dr}(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})\tilde{\mathbf{I}}-\left(\frac{dA_{3}}{dr}-\frac{2A_{3}}{r}\right)(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}})(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})\hat{\mathbf{n}}_{\mathrm{y}}\otimes\hat{\mathbf{r}}-$$
$$\frac{A_{3}}{r}(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{n}}_{\mathbf{x}})\hat{\mathbf{n}}_{\mathrm{y}}\otimes\hat{\mathbf{r}}-\frac{A_{3}}{r}(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}})\hat{\mathbf{n}}_{\mathrm{y}}\otimes\hat{\mathbf{n}}_{\mathbf{x}}-$$
$$\left(\frac{dA_{4}}{dr}-\frac{2A_{4}}{r}\right)(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}})(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_{\mathrm{y}}--\frac{A_{4}}{r}(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{n}}_{\mathbf{x}})\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_{\mathrm{y}}-$$
$$\left.\frac{A_{4}}{r}(\hat{\mathbf{n}}_{\mathrm{y}}\cdot\hat{\mathbf{r}})\hat{\mathbf{n}}_{\mathbf{x}}\otimes\hat{\mathbf{n}}_{\mathrm{y}}-\frac{dA_{5}}{dr}(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})\hat{\mathbf{n}}_{\mathrm{y}}\otimes\hat{\mathbf{n}}_{\mathrm{y}}\right] \quad \text{(II.10)}$$

$$
\begin{aligned}
&\frac{\partial}{\partial n_{\mathbf{x}}}\left[\left(\nabla_S \hat{\mathbf{n}}_{\mathbf{y}}\right):\widetilde{\widetilde{\boldsymbol{\mu}}}^{*}\right]^{T} = -\frac{g^2}{16\pi(1-\nu)}\Bigg[\Bigg[\left(\frac{dA_1}{dr}-\frac{4A_1}{r}\right)(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})\left(\nabla_S \hat{\mathbf{n}}_{\mathbf{y}}\right):\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}+ \\
&\frac{A_1}{r}\left(\hat{\mathbf{n}}_{\mathbf{x}}\otimes\hat{\mathbf{r}}+\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_{\mathbf{x}}\right):\left(\nabla_S \hat{\mathbf{n}}_{\mathbf{y}}\right)+\left(\frac{dA_6}{dr}-\frac{2A_6}{r}\right)(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})\left(\nabla_S \cdot\hat{\mathbf{n}}_{\mathbf{y}}\right)\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\Bigg]\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}+ \\
&\Bigg[\left(\frac{dA_2}{dr}-\frac{2A_2}{r}\right)(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})\left(\nabla_S \hat{\mathbf{n}}_{\mathbf{y}}\right):\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}+ \\
&\frac{A_2}{r}\left(\hat{\mathbf{n}}_{\mathbf{x}}\otimes\hat{\mathbf{r}}+\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_{\mathbf{x}}\right):\left(\nabla_S \hat{\mathbf{n}}_{\mathbf{y}}\right)+\frac{dA_7}{dr}\left(\nabla_S \cdot\hat{\mathbf{n}}_{\mathbf{y}}\right)\Bigg]\widetilde{\mathbf{I}}+ \\
&\left[\frac{A_6}{r}\left(\nabla_S \cdot\hat{\mathbf{n}}_{\mathbf{y}}\right)+\frac{A_1}{r}\left(\nabla_S \hat{\mathbf{n}}_{\mathbf{y}}\right):\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\right]\left(\hat{\mathbf{n}}_{\mathbf{x}}\otimes\hat{\mathbf{r}}+\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_{\mathbf{x}}\right)+ \\
&\left(\frac{dA_6}{dr}-\frac{2A_6}{r}\right)(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})\left[(\hat{\mathbf{r}}\otimes\hat{\mathbf{r}})\cdot\left(\nabla_S \hat{\mathbf{n}}_{\mathbf{y}}\right)+\hat{\mathbf{r}}\cdot\left(\nabla_S \hat{\mathbf{n}}_{\mathbf{y}}\right)\otimes\hat{\mathbf{r}}\right]+ \\
&\frac{A_6}{r}\left[\hat{\mathbf{n}}_{\mathbf{x}}\cdot\left(\nabla_S \hat{\mathbf{n}}_{\mathbf{y}}\right)\otimes\hat{\mathbf{r}}+\hat{\mathbf{r}}\cdot\left(\nabla_S \hat{\mathbf{n}}_{\mathbf{y}}\right)\otimes\hat{\mathbf{n}}_{\mathbf{x}}+\left(\hat{\mathbf{n}}_{\mathbf{x}}\otimes\hat{\mathbf{r}}+\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_{\mathbf{x}}\right)\cdot\left(\nabla_S \hat{\mathbf{n}}_{\mathbf{y}}\right)\right]+ \\
&\left(\frac{dA_2}{dr}-\frac{2A_2}{r}\right)(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})\left(\nabla_S \hat{\mathbf{n}}_{\mathbf{y}}\right)\cdot(\hat{\mathbf{r}}\otimes\hat{\mathbf{r}})+ \\
&\frac{A_2}{r}\left(\nabla_S \hat{\mathbf{n}}_{\mathbf{y}}\right)\cdot\left(\hat{\mathbf{n}}_{\mathbf{x}}\otimes\hat{\mathbf{r}}+\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_{\mathbf{x}}\right)+\frac{A_9}{r}\left(\hat{\mathbf{n}}_{\mathbf{x}}\otimes\hat{\mathbf{r}}+\hat{\mathbf{r}}\otimes\hat{\mathbf{n}}_{\mathbf{x}}\right)\cdot\left(\nabla_S \hat{\mathbf{n}}_{\mathbf{y}}\right)^T+ \\
&\left(\frac{dA_9}{dr}-\frac{2A_9}{r}\right)(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})(\hat{\mathbf{r}}\otimes\hat{\mathbf{r}})\cdot\left(\nabla_S \hat{\mathbf{n}}_{\mathbf{y}}\right)^T+ \\
&\frac{dA_8}{dr}(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})\left(\nabla_S \hat{\mathbf{n}}_{\mathbf{y}}\right)^T+\frac{dA_7}{dr}(\hat{\mathbf{n}}_{\mathbf{x}}\cdot\hat{\mathbf{r}})\left(\nabla_S \hat{\mathbf{n}}_{\mathbf{y}}\right)\Bigg]
\end{aligned}
\tag{II.11}
$$

$$
\begin{aligned}
&A = 2\left(\frac{2\mathrm{X}}{r}-\frac{d\mathrm{X}}{dr}\right), \quad B = \frac{d\Psi}{dr}-\frac{\mathrm{X}}{r}, \\
&C = \frac{2\nu}{1-2\nu}\left(\frac{d\Psi}{dr}-\frac{d\mathrm{X}}{dr}-\frac{2\mathrm{X}}{r}\right)-\frac{2\mathrm{X}}{r}
\end{aligned}
\tag{II.12}
$$

$$
\begin{aligned}
&A_1 = \frac{dA}{dr}-\frac{3A}{r},\ A_2 = \frac{dB}{dr}-\frac{B}{r},\ A_3 = \frac{dB}{dr}-\frac{B}{r}+\frac{A}{r},\ A_4 = \frac{dC}{dr}-\frac{C}{r}+\frac{A}{r}, \\
&A_5 = \frac{(B+C)}{r},\ A_6 = \frac{A}{r},\ A_7 = \frac{B}{r},\ A_8 = \frac{C}{r},\ A_9 = \frac{dC}{dr}-\frac{C}{r}
\end{aligned}
\tag{II.13}
$$

$$B_1 = \frac{d^2A}{dr^2} - \frac{7}{r}\frac{dA}{dr} + \frac{15A}{r^2},\ B_2 = \frac{3}{r}\frac{dA}{dr} - \frac{9A}{r^2},\ B_3 = \frac{d^2B}{dr^2} - \frac{3}{r}\frac{dB}{dr} + \frac{3B}{r^2},$$

$$B_4 = \frac{3}{r}\frac{dB}{dr} - \frac{3B}{r^2},\ B_5 = \frac{d^2C}{dr^2} - \frac{3}{r}\frac{dC}{dr} + \frac{3C}{r^2} + \frac{2}{r}\frac{dA}{dr} - \frac{6A}{r^2},$$

$$B_6 = \frac{2A}{r^2} + \frac{1}{r}\frac{dC}{dr} - \frac{C}{r^2}, \qquad \text{(II.14)}$$

$$B_7 = \frac{d^2B}{dr^2} - \frac{3}{r}\frac{dB}{dr} + \frac{3B}{r^2} + \frac{2}{r}\frac{dA}{dr} - \frac{6A}{r^2},\ B_8 = \frac{2A}{r^2} + \frac{1}{r}\frac{dB}{dr} - \frac{B}{r^2},$$

$$B_9 = \frac{2}{r}\frac{dB}{dr} - \frac{2B}{r^2} + \frac{2}{r}\frac{dC}{dr} - \frac{2C}{r^2} + \frac{2A}{r^2}$$

$$F_1 = \frac{d^2A}{dr^2} + \frac{2}{r}\frac{dA}{dr} - \frac{12A}{r^2},\ F_2 = \frac{d^2B}{dr^2} + \frac{2}{r}\frac{dB}{dr} - \frac{2B}{r^2} + \frac{2A}{r^2}, \qquad \text{(II.15)}$$

$$F_3 = \frac{d^2C}{dr^2} + \frac{2}{r}\frac{dC}{dr} - \frac{2C}{r^2} + \frac{2A}{r^2}$$

$$D_1 = \frac{\mathrm{d}^2A}{\mathrm{d}r^2} - \frac{6A}{r^2} + \frac{\mathrm{d}^2B}{\mathrm{d}r^2} - \frac{3}{r}\frac{\mathrm{d}B}{\mathrm{d}r} + \frac{3B}{r^2} + \frac{\mathrm{d}^2C}{\mathrm{d}r^2} - \frac{3}{r}\frac{\mathrm{d}C}{\mathrm{d}r} + \frac{3C}{r^2},$$

$$D_2 = \frac{\mathrm{d}^2B}{\mathrm{d}r^2} + \frac{3}{r}\frac{\mathrm{d}B}{\mathrm{d}r} - \frac{3B}{r^2} + \frac{1}{r}\frac{\mathrm{d}C}{\mathrm{d}r} - \frac{C}{r^2},$$

$$D_3 = \frac{1}{r}\frac{\mathrm{d}A}{\mathrm{d}r} + \frac{2A}{r^2} + \frac{1}{r}\frac{\mathrm{d}B}{\mathrm{d}r} - \frac{B}{r^2} + \frac{1}{r}\frac{\mathrm{d}C}{\mathrm{d}r} - \frac{C}{r^2}, \qquad \text{(II.16)}$$

$$D_4 = \frac{\mathrm{d}A}{\mathrm{d}r} + \frac{2A}{r} + \frac{\mathrm{d}B}{\mathrm{d}r} - \frac{B}{r} + \frac{\mathrm{d}C}{\mathrm{d}r} - \frac{C}{r},$$

$$D_5 = \frac{\mathrm{d}B}{\mathrm{d}r} + \frac{3B}{r} + \frac{C}{r}$$

References

[1] Mindlin, R. D. Microstructure in linear elasticity. *Arch. Rat. Mech. Anal.* **10**, 51-78, 1964.

[2] Mindlin, R. D. Second gradient of strain and surface tension in linear elasticity. *Int. J. Solids Struct.* **1**, 417-438, 1965.

[3] Aifantis, E. C. On the role of gradients in the localization of deformation and fracture. *Int. J. Engng. Sci.* **30**, 1279-1299, 1992.

[4] Altan, S. and Aifantis, E. C. On the structure of the mode III crack-tip in gradient elasticity. *Scripta Metall. Mater*. **26**, 319-324, 1992.

[5] Ru, C. Q. and Aifantis, E. C. A simple approach to solve boundary value problems in gradient elasticity. *Acta Mechanica* **101**, 59-68, 1993.

[6] Vardoulakis, I. and Sulem, J. *Bifurcation Analysis in Geomechanics.* Blackie/Chapman and Hall, London, 1995.

[7] Exadaktylos, G. E. and Vardoulakis, I. Microstructure in linear elasticity and scale effects: a reconsideration of basic rock mechanics and rock fracture mechanics. *Tectonophysics* **335**, 81-109, 2001.

[8] Exadaktylos, G., Vardoulakis, I. and Aifantis, E. C. Cracks in gradient elastic bodies with surface energy. *Int. J. Fracture* **79**, 107-119, 1996.

[9] Gutkin, M. Y. and Aifantis, E. C. Dislocations in the theory of gradient elasticity. *Scripta Materialia* **40**, 559-566, 1999.

[10] Exadaktylos, G. E. and Vardoulakis, I. Surface instability in gradient elasticity with surface energy. *Int. J. Solids Struct*. **35**, 2251-2281, 1998.

[11] Tsepoura, K. G., Papargyri-Beskou, S., Polyzos, D. and Beskos, D. E. Static and dynamic analysis of a gradient elastic bar in tension. *Arch. Appl. Mech*. **72**, 483-497, 2002a.

[12] Papargyri-Beskou, S., Tsepoura, K. G., Polyzos, D. and Beskos, D. E. Bending and stability analysis of gradient elastic beams. *Int. J. Solids Struct.* **40**, 385-400, 2003a.

[13] Altan, B. S., Evensen, H. and Aifantis, E. C. Longitudinal vibrations of a beam:a gradient elasticity approach. *Mech. Res. Comm.* **23**, 35-40, 1996.

[14] Tsepoura, K. G., Papargyri-Beskou, S. and Polyzos, D. A boundary element method for solving 3D static gradient elastic problems with surface energy. *Comput. Mech.* **29**, 361-381, 2002b.

[15] Chang, C. S. and Gao, J. Wave propagation in granular rod using high-gradient theory. *J. Engng. Mech., ASCE* **123**, 52-59, 1997.

[16] Georgiadis, H. G., Vardoulakis, I. and Lykotrafitis, G. Torsional surface waves in a gradient-elastic half-space. *Wave Motion* **31**, 333-348, 2000.

[17] Shu, J. Y., King, W. E. and Fleck, N. A. Finite elements for materials with strain gradient effects. *Int. J. Num. Meth. Engng* **44**, 373-391, 1999.

[18] Amanatidou, E. and Aravas, N. Mixed finite element formulations of strain-gradient elasticity problems. *Comp. Meth. Appl. Mech. Engng* **191**, 1723-1751, 2002.

[19] Papargyri-Beskou, S., Polyzos, D. and Beskos, D. E. Dynamic analysis of gradient elastic flexural beams. *Struct. Engng & Mech*., in press, 2003b.

[20] Dassios, G. and Lindell, I. V. On the Helmholtz Decomposition for Polyadics. *Quarterly of Applied Mathematics* LIX **4**, 787-796, 2001.

[21] Dominguez, J. *Boundary Elements in Dynamics.* Computational Mechanics Publications, Southampton Boston, 1994.

[22] Teneketzis Tenek, L. and Aifantis, E. C. A two-dimensional finite element implementation of a special form of gradient elasticity. *Comp. Model. Engng. Sci*. **3**, 731-741, 2002.