A New Formulation for Solution of Boundary Value Problems Using Domain-type Approximations and Local Integral Equations

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Abstract

The boundary and domain-type approximations are discussed in boundary integral equation formulations for solution of boundary value problems. A new approach is proposed with using a domain-type approximation of the primary field and collocation of boundary conditions at boundary nodes and local integral representation of the primary field at interior nodes. Two kinds of the domain-type approximation are utilized. The proposed method is illustrated on potential problems in two dimensions.

1. Introduction

A huge amount of literature is devoted to numerical solutions of boundary value problems. The principal difference between the finite element method (FEM) and boundary integral equation method (BIEM) is in reduction of dimensionality [1, 2]. It means that the solution at an arbitrary point can be found, if we know relevant boundary densities without having known the solution at any other interior point in case of BIEM. Such a formulation is called also the pure boundary formulation. The main advantages of the BIEM approach result from the necessity to discretize only the boundary of the analyzed domain. On the other hand, the numerical integrations require an enhanced attention including the use of various regularization techniques [3, 4] owing to singular kernels originating from the fundamental solutions of governing differential operators.

It is well known that a pure boundary formulation is restricted to problems when the fundamental solution is available. A lot of criticism has been raised against the boundary element method (BEM) that it is not applicable to non-linear problems and/or problems governed by the differential equations with variable coefficients due to material non-homogeneities. Nevertheless, the BIE have been applied also to solution of such problems with using the fundamental solutions of simpler differential operators and treating the domain integrals by using the finite elements. Such an approach is called boundary-domain formulation in contrast to the pure boundary formulation. The simultaneous use of both the boundary elements and domain cells is not quite consistent from the point of view of approximation conception.

In the last decade an offence has been taken at the use of finite size elements (both the domain cells and boundary elements) with polynomial interpolations, because of the necessity of mesh generation. A lot of effort has been devoted to the development of various mesh-free approximations and implementations of both the integral equations and variational approaches employed in FEM formulations. It seems that the efficiency

of the use of the finite size elements or mesh-free approximations is dependent on the character of the boundary value problem, if re-meshing is required or not.

In this paper, we remember the standard BIEM with including the boundary-domain formulations in order to point out the difference between the boundary and domain approximation conceptions.

Further, we discuss the domain-type approximations and the use of physical principles in computation of nodal unknowns. We consider two kinds of domain-type approximations, such as standard finite elements and the moving-least-square approximation.

Simple numerical examples are considered for illustration of the proposed formulation.

2. Standard Boundary Integral Equation Method

Standard BIEM is based on the use of the boundary integral equations and/or integral representations with the integrations on the global boundary of the analyzed domain. The global boundary integral equation is a constraint equation utilized as a relationship between relevant physically conjugated boundary densities which are approximated as independent quantities using boundary elements.

2.1 Pure boundary formulation. Boundary element method

For illustration it is sufficient to consider a simple potential problem governed by the Laplace differential equation

$$\nabla^2 u(x) = 0 \quad \text{in} \quad \Omega \tag{1}$$

supplemented with the Dirichlet and Neumann boundary conditions given as, respectively,

$$u(\eta) = \overline{u}(\eta) \quad \text{if} \quad \eta \in \partial\Omega_D$$

$$\frac{\partial u}{\partial n}(\eta) = \overline{q}(\eta) \quad \text{if} \quad \eta \in \partial\Omega_N \tag{2}$$

where $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ is the complete boundary of the domain Ω.

The fundamental solution of the Laplace operator is defined as the solution of the Poisson equation in the infinite space with the point source generator, i.e.

$$\nabla^2 G(x - y) = -\delta(x - y) \tag{3}$$

where $\delta(r)$ is the Dirac δ-function. It is well known that

$$G(r) = \begin{cases} -\dfrac{1}{2\pi}\ln\left(\dfrac{r}{r_o}\right) , & \text{for } 2-d \text{ problems} \\[3mm] \dfrac{1}{4\pi r} , & \text{for } 3-d \text{ problems} \end{cases} \tag{4}$$

in which r_o is an arbitrary constant.

   In view of eq. (1), we have the integral identity

$$\int_{\Omega} G(x-y)\nabla^2 u(x)d\Omega(x) = 0 \tag{5}$$

Making use of the Gauss divergence theorem repeatedly, one can derive the boundary integral representation of the potential field

$$\Delta(y)u(y) = \int_{\partial\Omega}\left[\frac{\partial u}{\partial n}(\eta)G(\eta-y) - u(\eta)\frac{\partial G(\eta-y)}{\partial n(\eta)}\right]d\Gamma(\eta) \tag{6}$$

where

$$\Delta(y) = \begin{cases} 1 , & y \in \Omega \\ 0 , & y \notin (\Omega \cup \partial\Omega) \end{cases} \tag{7}$$

Owing to the strong singularity of gradients of the fundamental solution, the integral of the second term behind the integral sign in eq. (6) is not continuous across the boundary [5, 6]. Without going into details [4], we remember that the Cauchy principal value (CPV) concept can be avoided by using the integral identity

$$\int_{\partial\Omega}\frac{\partial G(\eta-y)}{\partial n(\eta)}d\Gamma(\eta) = -\Delta(y) \tag{8}$$

which is an equivalent of the rigid body motion idea employed in elasticity [7]. Assuming the boundary density of the potential to be Hölder continuous on the Ljapunov boundary curve (or surface), we can perform the limit $\Omega \ni y \to \zeta \in \partial\Omega$ in eq. (6) with substituted eq. (8) directly by putting $y = \zeta$. Thus, we get the regularized boundary integral equation (BIE) with singular kernels

$$\int_{\partial\Omega}[u(\eta)-u(\zeta)]\frac{\partial G(\eta-\zeta)}{\partial n(\eta)}d\Gamma(\eta) - \int_{\partial\Omega}\frac{\partial u}{\partial n}(\eta)G(\eta-\zeta)d\Gamma(\eta) = 0 \tag{9}$$

According to the integral representation (6), one can evaluate the solution of the boundary value problem at each point in Ω without having known the solution at any other interior point. It is sufficient to know the distribution of both the boundary densities over the all boundary $\partial\Omega$. Thus, the formulation of the solution has a pure boundary character.

Since the distributions of two relevant boundary densities involve information about variations of the potential field in two orthogonal directions, these boundary densities are mutually independent in mathematical language and/or canonically conjugated in physical language. It means that the variation of one of them over a certain segment of the boundary is not sufficient for getting the variation of the conjugated quantity by a mathematical manipulation without incorporating the physical coupling. Thus, on the Dirichlet part of the boundary, one can obtain the tangent derivatives of the potential along the boundary by differentiation, but the evaluation of the normal derivative requires knowledge of the potentials along the normal direction. Similar, by integrating the normal derivatives of the potential along a line orthogonal to the boundary, one cannot get the boundary value of the potential without having known the value of the potential at a point on this line. The boundary integral equation (9) is an integral equivalent of the governing equation (1). Thus, it represents the needed physical coupling or plays the role of a physical constraint that should be satisfied by the mathematically independent boundary densities. Having solved the BIE supplemented with the prescribed boundary conditions, one can get the distributions of u and $\partial u / \partial n$ along $\partial\Omega_N$ and $\partial\Omega_D$, respectively.

It is natural to discretize the boundary into finite size elements in numerical solution of the BIE, in order to model the geometry of the boundary and to approximate the boundary densities of relevant physical quantities. The formulation of the solution of boundary value problems by boundary integral equations and their numerical implementation by boundary elements are two corner stones leading to remarkable efficiency and accuracy of the boundary integral equation method [1] and/or boundary element method [2] when applied to problems where pure boundary formulation is available.

2.2 Boundary-domain formulation

One of the key points in the derivation of a pure boundary formulation is finding the fundamental solution. Note that the closed form fundamental solutions are available only for relatively simple differential operators with constant coefficients. Of course, for solution of non-linear problems, one cannot find a pure boundary formulation. Sometimes the fundamental solution of the governing differential operator is not employed because of its complexity. Replacing the adequate fundamental solution by a simpler one, we cannot get a pure boundary formulation too. Then, the domain integrals of the unknown field are involved in the formulation in addition to the boundary ones.

For illustration, let us consider a potential problem in media with material non-homogeneity. The governing equation is given as

$$\left(k(x)u_{,i}(x)\right)_{,i} = f(x) \quad \text{in} \quad \Omega \tag{10}$$

According to one of the physical interpretations, $u(x)$ describes the stationary temperature field distribution with k and f being the heat conduction coefficient and body heat source density, respectively. The subscripts following a comma denote the partial derivatives with respect to Cartesian coordinates.

Assuming $k(x)$ to be a prescribed function of spatial coordinates, equation (10) can be rearranged as

$$\nabla^2 u(x) + \frac{k_{,i}(x)}{k(x)} u_{,i}(x) = \frac{f(x)}{k(x)} \tag{11}$$

For simplicity, let us consider the Dirichlet and Neumann boundary conditions given by eq. (2). Strictly speaking, the flux q is prescribed on $\partial\Omega_N$. However, the normal derivative can be easily expressed by $\partial u / \partial n = q / k$.

It should be stressed that the fundamental solution for the operator

$$\left(\nabla^2 + \frac{k_{,i}(x)}{k(x)} \partial_i \right)$$

is not available in closed form, in general. Nevertheless, one can formulate the solution of a boundary value problem using the integral equation approach.

Making use of the fundamental solution for the Laplace operator, we can recast the integral identity

$$\int_\Omega G(x-y)\nabla^2 u(x)d\Omega(x) = \int_\Omega \frac{G(x-y)}{k(x)}\Big[f(x) - k_{,j}(x)u_{,j}(x) \Big]d\Omega(x) \tag{12}$$

into the integral representation of the potential field

$$\Delta(y)u(y) = \int_{\partial\Omega}\left[\frac{\partial u}{\partial n}(\eta)G(\eta-y) - u(\eta)\frac{\partial G(\eta-y)}{\partial n(\eta)} \right]d\Gamma(\eta) -$$

$$-\int_\Omega \frac{G(x-y)}{k(x)}\Big[f(x) - k_{,j}(x)u_{,j}(x) \Big]d\Omega(x) \tag{13}$$

Owing to the domain integral of unknown potential gradients, the formulation has not any more the pure character. Thus, it is not sufficient to solve the system of the BIE for boundary unknowns, but one has to discretize also the interior of the domain in order to compute simultaneously the potential gradients. The integral representation of the potential gradients can be obtained directly by differentiating eq. (13)

$$\Delta(y)u_{,i}(y) = \int_{\partial\Omega}\left[u(\eta)n_j(\eta)G_{,ij}(\eta-y) - \frac{\partial u}{\partial n}(\eta)G_{,i}(\eta-y) \right]d\Gamma(\eta) +$$

$$+\int_\Omega \frac{G_{,i}(x-y)}{k(x)}\Big[f(x) - k_{,j}(x)u_{,j}(x) \Big]d\Omega(x) \tag{14}$$

Besides this integral representation with hypersingular kernel, one can derive also the partially regularized integral representation with strongly singular kernels starting from the integral identity

$$\int_{\Omega} G_{,i}(x-y)\nabla^2 u(x)d\Omega(x) = \int_{\Omega} \frac{G_{,i}(x-y)}{k(x)}\Big[f(x)-k_{,j}(x)u_{,j}(x)\Big]d\Omega(x) \qquad (15)$$

Hence [4],

$$\Delta(y)u_{,i}(y) = \int_{\partial\Omega}\left[\overset{*}{\varepsilon}_{kij}\hat{D}_k u(\eta)-\delta_{ij}\frac{\partial u}{\partial n}(\eta)\right]G_{,j}(\eta-y)d\Gamma(\eta) +$$

$$+\int_{\Omega}\frac{G_{,i}(x-y)}{k(x)}\Big[f(x)-k_{,j}(x)u_{,j}(x)\Big]d\Omega(x) \qquad (16)$$

where

$$\overset{*}{\varepsilon}_{kij} = \begin{cases} \delta_{k3}\varepsilon_{3ij} & , \quad \text{2-d problems} \\ \varepsilon_{kij} & , \qquad \text{3-d problems} \end{cases}$$

and the differential operator $\hat{D}_k$ is expressed in terms of the tangential derivatives as

$$\hat{D}_k = \rho_k(\eta)\frac{\partial}{\partial\tau(\eta)} - \tau_k(\eta)\frac{\partial}{\partial\rho(\eta)} = E_{kl}(\eta)\frac{\partial}{\partial\eta_l} \quad , \quad E_{kl} = \rho_k\tau_l - \tau_k\rho_l$$

with $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ being the unit tangent vectors related to the outward normal vector $\mathbf{n}$ by $\boldsymbol{\rho} = \mathbf{n}\times\boldsymbol{\tau}$. In the case of two-dimensional problems $\rho_k = \delta_{k3}$, hence, $n_3 = \tau_3 = 0$ and $\partial/\partial\rho = 0$, since all the quantities are independent on the third coordinate.

Recall that the tangent derivatives of the potential in eq. (16) do not give rise to new boundary unknowns, because they can be obtained from the approximation of the potential within a boundary element by direct differentiation with respect to intrinsic coordinates.

The regularized BIE resulting from eq. (13) is given by

$$\int_{\partial\Omega}\big[u(\eta)-u(\zeta)\big]\frac{\partial G(\eta-\zeta)}{\partial n(\eta)}d\Gamma(\eta) - \int_{\partial\Omega}\frac{\partial u}{\partial n}(\eta)G(\eta-y)d\Gamma(\eta) +$$

$$+\int_{\Omega}u_{,j}(x)\frac{k_{,j}(x)}{k(x)}G(x-\zeta)d\Omega(x) = \int_{\Omega}\frac{f(x)}{k(x)}G(x-\zeta)d\Omega(x) \qquad (17)$$

Performing the discretization of the boundary into boundary elements and domain into cells, one can compute the unknown nodal values on the boundary and potential gradients at interior nodes by solving the system of the discretized BIE (17) and one of

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the integral representations given by eqs. (14) and (16). It is not necessary to consider the integral representation of the potential gradients at boundary nodes, since they can be expressed in terms of the normal and tangential derivatives as

$$u_{,i}(\zeta) = n_i(\zeta)\frac{\partial u}{\partial n}(\zeta) + \tau_i(\zeta)\frac{\partial u}{\partial \tau}(\zeta) + \rho_i(\zeta)\frac{\partial u}{\partial \rho}(\zeta) \tag{18}$$

Thus, both the strongly- and hyper-singular integral representations are applicable without further regularization when collocating at interior nodes.

Similar boundary-domain formulation can be derived also for solution of the boundary value problems with non-linear constitutive law and/or for time-dependent problems with using static fundamental solutions in order to avoid the complexity of the time-dependent ones. Undoubtedly, the boundary domain formulation is much less effective than a pure boundary formulation because of the necessity to discretize also the interior of the domain. Nevertheless, it works reliably [8]. As compared with the FEM formulation, both the formulations require domain elements, but the mathematics is more complex in the boundary-domain formulation of the integral equation approach. The singular kernels in the integral equation approach lead to a localization yielding better conditioning of the discretized system of equations, while the polynomial trial functions in the FEM approach result in smoothing effects. Despite the boundary-domain formulation is operating, one can point out certain discrepancy between two conceptions of the approximation of boundary densities in the BIE and the approximation of fields over domain-type elements.

3. Domain-type approximations

Contrary to boundary elements, the dimensionality of the problem is not reduced if approximations within domain elements are involved. It should be pointed out that approximating a field within a domain element, one can get complete gradients of that field by differentiating the approximation of the primary field. Thus, the conception of independent approximations of boundary densities employed in discretization of the BIE is not consistent with the conception of domain-type approximation of the same primary field as proposed in the boundary-domain formulation of the integral equation approach.

Let us consider a simple potential problem governed by the Laplace differential equation in a square $L \times L$ with the boundary conditions shown in Fig.1.
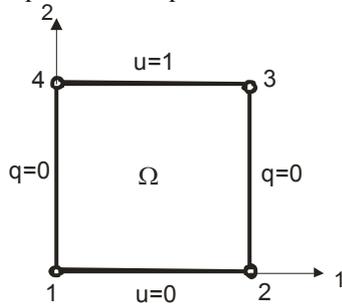

Fig. 1 Sketch of the b.v.p.

Obviously, the exact solution $u^{ex}(x) = x_2/L$ can be fitted by using a linear interpolation in the x_2-direction within the square element. Thus, using the bilinear quadrilateral Lagrange element S_1 with nodal points at corners of the square domain Ω,

$$u(x)\big|_{S_1} = \sum_{a=1}^{4} u(x^{a1}) N^a(\xi_{1,}\xi_2),$$

x_i^{a1} - Cartesian coordinates of the a-th nodal point on S_1,

we obtain the exact solution by collocating only the Dirichlet boundary conditions, since the potential is prescribed at each node x^a $(a = 1,2,3,4)$.

The shape functions and their derivatives for the bilinear quadrilateral Lagrange element are given in Table 1.

Table 1

	$a = 1$	$a = 2$	$a = 3$	$a = 4$
$N^a(\xi_1,\xi_2)$	$\dfrac{1}{4}(1-\xi_1)(1-\xi_2)$	$\dfrac{1}{4}(1+\xi_1)(1-\xi_2)$	$\dfrac{1}{4}(1+\xi_1)(1+\xi_2)$	$\dfrac{1}{4}(1-\xi_1)(1+\xi_2)$
$N_{,1}^a(\xi_1,\xi_2)$	$\dfrac{1}{4}(\xi_2-1)$	$\dfrac{1}{4}(1-\xi_2)$	$\dfrac{1}{4}(1+\xi_2)$	$-\dfrac{1}{4}(1+\xi_2)$
$N_{,2}^a(\xi_1,\xi_2)$	$\dfrac{1}{4}(\xi_1-1)$	$-\dfrac{1}{4}(1+\xi_1)$	$\dfrac{1}{4}(1+\xi_1)$	$\dfrac{1}{4}(1-\xi_1)$

with $\xi_\alpha \in [-1,1]$. Hence, the bilinear interpolation on $\Omega = S_1$ yields

$$u(x)\big|_{S_1} = \sum_{a=1}^{4} u^a N^a(\xi_{1,}\xi_2) = N^3(\xi_{1,}\xi_2) + N^4(\xi_{1,}\xi_2) = \frac{1+\xi_2}{2} = \frac{x_2}{L}$$

and $\quad \dfrac{\partial u}{\partial x_i} = \dfrac{\delta_{i2}}{L}$.

Thus, the exact solution is reproduced.

Of course, in general, it is insufficient to assume a linear variation of the solution and the subdivision of the analyzed domain into more elements is required, $\Omega = \bigcup\limits_{e=1}^{m} S_e$.

If we use four bilinear quadrilateral Lagrange elements as shown in Fig.2 for the same b.v.p., we may write six equations by collocating the Dirichlet boundary conditions at global nodes x^1, x^2, x^5, x^7, x^8, and x^9 as

$$u(x^1) = \sum_{a=1}^{4} u(x^{a1}) N^a(-1,-1) = u^1 = 0 \, , \qquad u(x^2) = \sum_{a=1}^{4} u(x^{a1}) N^a(1,-1) = u^2 = 0$$

$$u(x^5) = \sum_{a=1}^{4} u(x^{a2}) N^a(1,-1) = u^5 = 0 \, , \qquad u(x^7) = \sum_{a=1}^{4} u(x^{a3}) N^a(1,1) = u^7 = 1$$

$$u(x^8) = \sum_{a=1}^{4} u(x^{a3}) N^a(-1,1) = u^8 = 1 \, , \qquad u(x^9) = \sum_{a=1}^{4} u(x^{a4}) N^a(-1,1) = u^9 = 1$$
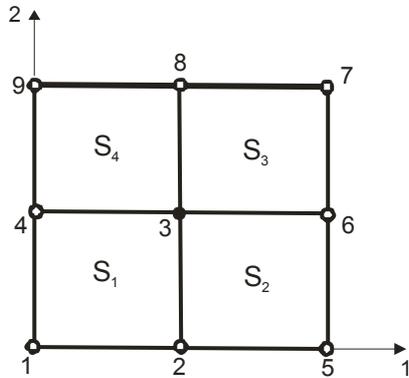
$$(19)$$

Fig. 2 Discretization into 4 Lagrange elements $\left(\Omega = S_1 \cup S_2 \cup S_3 \cup S_4 \right)$

In order to collocate the Neumann boundary conditions, we need to express the potential gradients on domain elements in terms of the partial derivatives with respect to intrinsic coordinates. Assuming the isoparametric approximations, we have

$$u(x)\big|_{S_e} = \sum_{a=1}^{n} u(x^{ae}) N^a(\xi_1, \xi_2) \tag{20}$$

$$x_i\big|_{S_e} = \sum_{a=1}^{n} x_i^{ae} N^a(\xi_1, \xi_2) \tag{21}$$

Hence,

$$\frac{\partial(\bullet)}{\partial x_i}\bigg|_{S_e} = \frac{\partial \xi_j}{\partial x_i} \frac{\partial(\bullet)}{\partial \xi_j}\bigg|_{S_e} = \left(h^e \right)_{ij}^{-1} \frac{\partial(\bullet)}{\partial \xi_j}\bigg|_{S_e} \tag{22}$$

where $\left(h^e \right)^{-1}$ is the inverse matrix to $h^e_{..}$ defined as

$$h_{\alpha i}^{e}(\xi_{1,}\xi_{2}) = \sum_{a=1}^{n} x_{i}^{ae} N_{,\alpha}^{a}(\xi_{1,}\xi_{2}) \tag{23}$$

Since

$$\left(h^{e}\right)_{ij}^{-1} = \frac{\varepsilon_{3il}\varepsilon_{3jk}h_{kl}^{e}(\xi_{1},\xi_{2})}{\varepsilon_{3mn}h_{1m}^{e}(\xi_{1},\xi_{2})h_{2n}^{e}(\xi_{1},\xi_{2})} \tag{24}$$

we may write

$$\left.\frac{\partial u(x)}{\partial x_{i}}\right|_{S_{e}} = \frac{\varepsilon_{3il}\varepsilon_{3jk}h_{kl}^{e}(\xi_{1},\xi_{2})}{\varepsilon_{3mn}h_{1m}^{e}(\xi_{1},\xi_{2})h_{2n}^{e}(\xi_{1},\xi_{2})} \sum_{a=1}^{n} u(x^{ae}) N_{,j}^{a}(\xi_{1},\xi_{2}) \tag{25}$$

It can be seen that on each element shown in Fig. 2, we have

$$h_{\alpha i}^{e}(\xi_{1,}\xi_{2}) = \frac{L}{4}\delta_{i\alpha} \quad, \tag{26}$$

hence,

$$\left(h^{e}\right)_{ij}^{-1} = \frac{4}{L}\left(\delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2}\right) = \frac{4}{L}\delta_{ik}\delta_{jk} = \frac{4}{L}\delta_{ij} \tag{27}$$

and

$$\left.\frac{\partial u(x)}{\partial x_{i}}\right|_{S_{e}} = \frac{4}{L}\sum_{a=1}^{4} u(x^{ae}) N_{,i}^{a}(\xi_{1},\xi_{2}) \tag{28}$$

Denoting by Γ_{ab} the side of an element between the global nodes x^{a} and x^{b}, we may write

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma_{56}} = \left.\frac{\partial u}{\partial x_{1}}\right|_{\Gamma_{56}} = \frac{4}{L}\sum_{a=1}^{4} u(x^{a2}) N_{,1}^{a}(1,\xi_{2}) = \frac{\xi_{2}+1}{L}\left(u^{6} - u^{3}\right)$$

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma_{67}} = \left.\frac{\partial u}{\partial x_{1}}\right|_{\Gamma_{67}} = \frac{\xi_{2}-1}{L}\left(u^{3} - u^{6}\right)$$

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma_{94}} = -\left.\frac{\partial u}{\partial x_{1}}\right|_{\Gamma_{94}} = \frac{1-\xi_{2}}{L}\left(u^{4} - u^{3}\right) \;, \quad \left.\frac{\partial u}{\partial n}\right|_{\Gamma_{41}} = -\left.\frac{\partial u}{\partial x_{1}}\right|_{\Gamma_{41}} = -\frac{\xi_{2}+1}{L}\left(u^{3} - u^{4}\right)$$

Bearing in mind the Neumann boundary conditions on $\Gamma_{56}, \Gamma_{67}, \Gamma_{94}$ and Γ_{41} we obtain two equations

$$u^{6} = u^{3} \;, \quad u^{4} = u^{3} \tag{29}$$

Thus, in this discretization, we get eight equations given by (19) and (29) when collocating the boundary conditions. Since the number of nodal unknowns is equal to nine, we need one more equation. The employed discretization and approximation is currently used in FEM formulation, where the complete system of algebraic equations for unknown potentials at nodal points is constructed as a weak formulation of the b.v.p. with using the approximation functions as the trial functions. Now, we propose to use the local integral equations (LIE) collocated at interior nodal points in order to complete the system of algebraic equations involving the collocation of the prescribed boundary conditions.

3.1 Local integral equations and their implementation by using finite elements (LIEFE)

Let S^c denote the union of all domain elements S_e adjacent to the interior nodal point x^c, i.e.,

$$S^c = \bigcup_{\substack{e=1 \\ x^c \in S_e}}^{m} S_e \quad , \text{with} \quad \Omega = \bigcup_{e=1}^{m} S_e \tag{30}$$

3.1.1 Potential problems governed by the Laplace differential equation (2-d)

Selecting a subdomain $\Omega^c \subset S^c$, we may write the local integral representation of the potential at x^c in terms of the potential and its normal derivative on the boundary $\partial \Omega^c$ as

$$u(x^c) = \int_{\partial \Omega^c} \left[\frac{\partial u}{\partial n}(\eta) G(\eta - x^c) - u(\eta) \frac{\partial G(\eta - x^c)}{\partial n(\eta)} \right] d\Gamma(\eta) \tag{31}$$

Let us decompose Ω^c and $\partial \Omega^c$ as

$$\Omega^c = \bigcup_{\substack{e=1 \\ x^c \in S_e}}^{m} \Omega_e^c \quad , \qquad \partial \Omega^c = \bigcup_{\substack{e=1 \\ x^c \in S_e}}^{m} \Gamma_e^c \tag{32}$$

with $\Omega_e^c = \Omega^c \cap S_e$ and $\Gamma_e^c = \partial \Omega^c \cap S_e$ for $S_e \in \left\{ \forall S_e ; \ x^c \in S_e \right\}$.

Assuming the finite element approximations and mapping Γ_e^c into L_e^c in the space of intrinsic coordinates, we obtain the discretized integral representation of the potential at x^c in terms of the nodal values of the potential on S^c as

$$u(x^c) = \sum_{\substack{e=1 \\ x^c \in S_e}}^{m} \sum_{a=1}^{n} u(x^{ae}) \int_{s_1^{Le}}^{s_2^{Le}} n_i \left(\eta(\xi^{Le}) \right) \Big[-N^a(\xi_1^{Le}, \xi_2^{Le}) G_{,i} \left(\eta(\xi^{Le}) - x^c \right) +$$

$$+ \left(h^e \right)_{ij}^{-1} (\xi_1^{Le}, \xi_2^{Le}) N_{,j}^{a} (\xi_1^{Le}, \xi_2^{Le}) G \left(\eta(\xi^{Le}) - x^c \right) \Big] h^e(s) ds \qquad (33)$$

where we utilized a parametrization of the contour segments L_e^c in intrinsic space

$$L_e^c = \left\{ \forall \boldsymbol{\xi}^{Le} \in R^2 ; \ \xi_k^{Le} = \xi_k^{Le}(s), \ s \in [s_1^{Le}, s_2^{Le}] \right\} \qquad (34)$$

Then, the Cartesian coordinates of $\boldsymbol{\eta} \in \Gamma_e^c$ are defined uniquely by the transformation

$$\eta_k(\xi^{Le}) = \sum_{a=1}^{n} x_k^{ae} N^a \left(\xi_1^{Le}(s), \ \xi_2^{Le}(s) \right) \qquad (35)$$

with the Jacobian of the transformation being given by

$$h^e(s) = \sqrt{h_k^e(s) h_k^e(s)} \ , \quad h_k^e(s) = \frac{\partial \eta_k^{Le}}{\partial s} = \frac{d\xi_\alpha^{Le}}{ds} \sum_{a=1}^{n} x_k^{ae} N_{,\alpha}^{a} \left(\xi_1^{Le}(s), \ \xi_2^{Le}(s) \right) \quad (36)$$

Since each discretized LIE brings coupling only among the nodes of the domain elements adjacent to the collocation point (interior node), the system matrix will not be fully populated like in FEM and contrary to standard BEM based on global BIE. Moreover, the LIE is nonsingular despite the singular kernels are employed.

Now, we can use the proposed LIEFE approach in solving the b.v.p. shown in Fig.2, where the bilinear quadrilateral Lagrange elements are employed. Since all the domain elements are adjacent to the interior node x^3, the discretized LIE involves all the nodal points independently of the choice of the subdomain Ω^c. Thus, one can take $\Omega = \Omega^c$, with $\partial\Omega^c$ being the global boundary $\partial\Omega$. Nevertheless, this approach differs from the BEM by the approximation. Taking $\partial\Omega^3 = \partial\Omega$, one can perform all the integrations in the discretized LIE analytically with the results

$$\int_{\partial\Omega^3} u(\eta) \frac{\partial G(\eta - x^3)}{\partial n(\eta)} d\Gamma(\eta) = -\frac{1}{2\pi} \Big\{ \left(u^1 - u^2 + u^5 - u^6 + u^7 - u^8 + u^9 - u^4 \right) \ln 2 +$$

$$+ \frac{\pi}{2} \left(u^2 + u^6 + u^8 + u^4 \right) \Big\}$$

$$\int_{\partial\Omega^3} \frac{\partial u}{\partial n}(\eta)G(\eta - x^3)d\Gamma(\eta) = -\frac{1}{2\pi}\left\{-\frac{1}{2}\left(u^1 - u^2 + u^5 - u^6 + u^7 - u^8 + u^9 - u^4\right) + \right.$$

$$\left. +\left(u^2 + u^6 + u^8 + u^4 - 4u^3\right)\left[\frac{\pi}{2} - \ln 2 - \frac{3}{2}\right]\right\}$$

Note that we have used $r_o = L$ for the calibration of the fundamental solution given by eq. (4). Hence, and from (19) and (29), the LIE

$$u^3 = \int_{\partial\Omega^3}\left[\frac{\partial u}{\partial n}(\eta)G(\eta - x^3) - u(\eta)\frac{\partial G(\eta - x^3)}{\partial n(\eta)}\right]d\Gamma(\eta)$$

becomes

$$2\left[\ln 2 + 1\right](1 - 2u^3) = 0 \qquad\qquad (37)$$

Thus, in view of (35) and (29), we have

$$u^3 = u^4 = u^6 = \frac{1}{2} \qquad\qquad (38)$$

The bilinear interpolations within four finite elements (Fig. 2) supplemented with the calculated values according to (19) and (38) result in the exact solution of the considered b.v.p.

Of course , any finer subdivision should reproduce the exact solution too. In order to demonstrate the sparsity of the system matrix, we have also applied the LIEFE to the same b.v.p. with using nine bilinear quadrilateral Lagrange elements with 16 nodal points as shown in Fig.3.
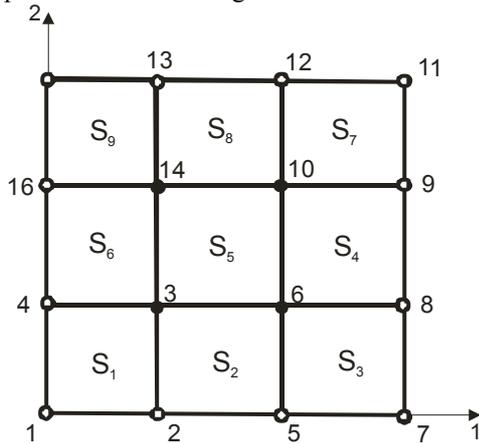

Fig. 3 Discretization into 9 Lagrange elements

Now, in view of eqs. (20)-(25), we have

$$h^e_{\alpha i}(\xi_1, \xi_2) = \frac{L}{6}\delta_{i\alpha} \ , \ \left(h^e\right)^{-1}_{ij} = \frac{6}{L}\delta_{ij} \ , \ \left.\frac{\partial u(x)}{\partial x_i}\right|_{S_e} = \frac{6}{L}\sum_{a=1}^{4} u(x^{ae})N^a_{,i}(\xi_1, \xi_2) \quad (39)$$

The Dirichlet boundary conditions give directly the nodal values

$$u^1 = u^2 = u^5 = u^7 = 0 \ , \qquad u^{11} = u^{12} = u^{13} = u^{15} = 1 \tag{40}$$

According to eq. (39), the Neumann boundary conditions yield

$$u^8 = u^6 \ , \qquad u^9 = u^{10} , \qquad u^{16} = u^{14} \ , \qquad u^4 = u^3 \tag{41}$$

To complete the system of algebraic equations for nodal potentials, we propose to collocate the LIE at the interior nodes x^3, x^6, x^{10} and x^{14}. If we select the four adjacent elements to an interior nodal point x^c as the subdomain Ω^c on which the LIE is considered, we can perform the adequate integrations over $\partial\Omega^c$ analytically. Then, taking eqs. (40) and (41) into account, we obtain the discretized LIE on $\Omega^3, \Omega^6, \Omega^{10}$ and Ω^{14}, respectively, given as

$$Au^3 + u^6 - Bu^{10} - Cu^{14} = 0 \tag{42}$$

$$u^3 + Au^6 - Cu^{10} - Bu^{14} = 0 \tag{43}$$

$$-Bu^3 - Cu^6 + Au^{10} + u^{14} = 2B - 1 \tag{44}$$

$$-Cu^3 - Bu^6 + u^{10} + Au^{14} = 2B - 1 \tag{45}$$

in which we have used the notations

$$A = -4\ln 3 - 5 \ , \quad B = -\ln 3 - 1/2 \ , \quad C = B - 1 .$$

Thus, only four nodal values are coupled in the system of algebraic equations. Having solved the subsystem (42)-(45), we obtain

$$u^3 = u^6 = 1/3 \ , \qquad u^{10} = u^{14} = 2/3 \tag{46}$$

Hence, and from eq. (41), we receive

$$u^4 = u^8 = 1/3 \ , \qquad u^9 = u^{16} = 2/3 \tag{47}$$

The numerical results for the nodal unknowns are identical with the exact values. Then, the bilinear interpolations within the employed finite elements follow the exact solution given by linear distribution of the potential in x_2-direction.

Of course, the use of finite elements with higher order interpolation polynomials should yield exact solution provided that the integrations are exact.

3.1.2 Potential problems in non-homogeneous media

Let us consider the potential problem governed by eq. (10), which can be equivalently rewritten into the form given by eq. (11). The analyzed domain Ω is assumed to be subdivided into finite elements $\Omega = \bigcup_{e=1}^{m} S_e$ with the polynomial interpolation of the variation of the potential, potential gradients and isoparametric approximation of the geometry as given by eqs. (20)-(25).

The system of algebraic equations for computation of nodal values of the potential is composed of two subsystems. One of these subsystems is created by collocating the prescribed boundary conditions at the nodal points $\zeta^b \in \partial\Omega$ as

$$u^b = \overline{u}(\zeta^b), \quad \text{if} \quad \zeta^b \in \partial\Omega_D \tag{48}$$

$$\frac{1}{m^b} \sum_{\substack{e=1 \\ \zeta^b \in S_e}}^{m} (h^e)_{ij}^{-1}(\xi_1^{0e}, \xi_2^{0e}) \sum_{a=1}^{n} u(x^{ae}) N_{,j}^{a}(\xi_1^{0e}, \xi_2^{0e}) \lim_{\partial\Omega_{Nb} \ni \eta \to \zeta^b} n_i(\eta) =$$

$$= \frac{1}{k(\zeta^b)} \lim_{\partial\Omega_{Nb} \ni \eta \to \zeta^b} \overline{q}(\eta), \quad \text{if} \quad \zeta^b \in \left(\partial\Omega_N - \partial\Omega_D\right) \tag{49}$$

in which (ξ_1^{0e}, ξ_2^{0e}) are the intrinsic coordinates of $\zeta^b \in S_e$, m^b is the number of finite elements adjacent to ζ^b, and $\partial\Omega_{Nb} = \partial\Omega_N \cap \partial S_{Eb}$, where S_{Eb} is one of the finite elements adjacent to ζ^b, i.e., $S_{Eb} \in \left\{\forall S_e \; ; \; \zeta^b \in S_e\right\}$.

The rest of the algebraic equations will be obtained by discretization of local integral representations of the potential field at interior nodes. According to eq. (13), we may write on Ω^c

$$u(x^c) = \int_{\partial\Omega^c} \left[\frac{\partial u}{\partial n}(\eta) G(\eta - x^c) - u(\eta) \frac{\partial G(\eta - x^c)}{\partial n(\eta)} \right] d\Gamma(\eta) +$$

$$+ \int_{\Omega^c} \frac{k_{,j}(x)}{k(x)} u_{,j}(x) G(x - x^c) d\Omega(x) + F^c \tag{50}$$

where

$$F^c = -\int_{\Omega^c} \frac{f(x)}{k(x)} G(x - x^c) d\Omega(x)$$

Now, in view of the finite element approximations (20)-(25), geometrical decompositions (32), and mapping Γ_e^c and Ω_e^c into L_e^c and D_e^c, respectively, in the space of intrinsic coordinates (ξ_1, ξ_2), we obtain the discretized LIE from (50)

$$u(x^c) = \sum_{\substack{e=1 \\ x^c \in S_e}}^{m} \sum_{a=1}^{n} u(x^{ae}) \left\{ \int_{L_e^c} n_i\left(\eta(\xi)\right) \left[\left(h^e\right)_{ij}^{-1}(\xi_1,\xi_2) N_{,j}^a(\xi_1,\xi_2) G\left(\eta(\xi)-x^c\right) - \right. \right.$$

$$\left. - N^a(\xi_1,\xi_2) G_{,i}\left(\eta(\xi)-x^c\right) \right] dL(\xi) +$$

$$\left. + \int_{D_e^c} \frac{k_{,i}\left(x(\xi)\right)}{k\left(x(\xi)\right)} \left(h^e\right)_{ij}^{-1}(\xi_1,\xi_2) N_{,j}^a(\xi_1,\xi_2) G\left(x(\xi)-x^c\right) J^e(\xi_1,\xi_2) d\xi_1 d\xi_2 \right\} + F^c$$

(51)

where

$$\eta_k\big|_{L_e^c} = \sum_{a=1}^{n} x_k^{ae} N^a(\xi_1,\xi_2) \quad \text{with} \quad (\xi_1,\xi_2) \in L_e^c$$

$$x_k\big|_{D_e^c} = \sum_{a=1}^{n} x_k^{ae} N^a(\xi_1,\xi_2) \quad \text{with} \quad (\xi_1,\xi_2) \in D_e^c$$

and $J^e(\xi_1,\xi_2)$ is the Jacobian of the transformation $(x_1,x_2) \to (\xi_1,\xi_2)$ and it is given by

$$J^e(\xi_1,\xi_2) = \left| \varepsilon_{3ij} h_{1i}^e(\xi_1,\xi_2) h_{2j}^e(\xi_1,\xi_2) \right|$$

(52)

Finally, making use of the parametrization (34), we may replace the contour integral in (51) by

$$\int_{L_e^c} (\bullet)\left(\eta(\xi_1,\xi_2)\right) dL(\xi) = \int_{s_1^{Le}}^{s_2^{Le}} (\bullet)\left(\eta(\xi_1^{Le}(s),\xi_2^{Le}(s))\right) h^e(s) ds$$

(53)

Example

Let us consider the b.v.p. governed by the equation

$$\nabla^2 u(x) + \frac{k_{,i}(x)}{k(x)} u_{,i}(x) = \frac{f(x)}{k(x)} \quad \text{on a square } L \times L$$

(54)

with the material non-homogeneity given as

$$k(x) = k_o e^{\delta x_2 / L}$$

(55)

and the boundary conditions are shown in Fig. 1.
    The exact solution is given by

$$u(x) = \frac{1 - e^{-\delta x_2 / L}}{1 - e^{-\delta}}$$

(56)

In order to get a faithful approximation of the potential interpolations, one should use several elements in x_2-direction due to the variation of the material parameters. Since

the boundary integrals in the LIE for the formulation of the solution of the b.v.p. in a non-homogeneous medium are the same as in the homogeneous one, we apply the discretization shown in Fig. 3. Then, the Dirichlet and Neumann boundary conditions result in eqs. (40) and (41), respectively.

In order to construct the LIE on the subdomain $\Omega^c = S^c$, one has to perform the integrations over the finite elements adjacent to the interior node x^c. In this case, all the integrals can be found in a closed form and the system of the LIE on $\Omega^3, \Omega^6, \Omega^{10}$, and Ω^{14} is given as

$$Au^3 + u^6 - (B+D)u^{10} - (B-1+E)u^{14} = 0 \tag{57}$$

$$u^3 + Au^6 - (B-1+E)u^{10} - (B+D)u^{14} = 0 \tag{58}$$

$$-(B-D)u^3 - (B-1-E)u^6 + Au^{10} + u^{14} = 2B - 1 + D + E \tag{59}$$

$$-(B-1-E)u^3 - (B-D)u^6 + u^{10} + Au^{14} = 2B - 1 + D + E \tag{60}$$

in which

$$D = \frac{\delta}{18}\left(2\ln 2 - 3\ln 3 - \frac{7}{2} + \frac{\pi}{2}\right) , \qquad E = \frac{\delta}{18}\left(4\ln 2 - 9\ln 3 - \frac{29}{2} + \frac{5}{2}\pi\right) \tag{61}$$

and the constants A, B have been defined above.

Having solved the system of equations (57)-(60) for nodal unknowns, we receive

$$u^{14} = u^{10} , \quad u^6 = u^3 , \quad u^3 = \frac{(\alpha + \beta)^2}{3\alpha^2 + \beta^2} , \quad u^{10} = \frac{2\alpha(\alpha + \beta)}{3\alpha^2 + \beta^2} \tag{62}$$

where

$$\alpha = 1 - 2B = 2(1 + \ln 3) , \qquad \beta = -(D+E) = \frac{\delta}{3}\left(3 + 2\ln 3 - \ln 2 - \frac{\pi}{2}\right)$$

The increase of the material parameter k at $x_2 = L$ and the accuracy of the numerically computed potential at interior nodes are shown in Table 2.

Table 2 Numerical results in non-homogeneous medium

δ	$k(x_2 = L) = k_o e^\delta$	% err $u(x_2 = L/3)$	% err $u(x_2 = 2L/3)$
0.1	1.105170918	1.2996	0.6432
0.01	1.010050167	0.1323	0.0661

Since all the integrations have been carried out analytically, the errors are given by the discretization and polynomial interpolation of the potential field in the numerical

computation. Of course, three elements in x_2-direction would not be enough for higher values of the parameter δ.

3.2 MLS-approximation in combination with Local Integral Equations

Another of the domain-type approximations is the Moving Least Square (MLS)-approximation, in which no elements are created. The nodal points are scattered freely throughout the analyzed domain and the contribution of particular nodes into the final approximation is controlled by certain weight functions employed in the determination of the coefficients in polynomial expansions by using weighted least-square-method. The approximant $u^h(x)$ of the potential field is defined by

$$u^h(x) = p_\alpha(x)c_\alpha(x) \qquad \text{at} \quad x \in \Omega_x \tag{63}$$

where $\{p_\alpha(x)\}_{\alpha=1}^{\mu}$ is a complete monomial basis of order μ, and $c_\alpha(x)$ are the expansion coefficients defined at each point x and determined by minimizing the weighted least-square functional

$$J(x) = \sum_{a=1}^{N} H\left(w^a(x)\right)w^a(x)\left[p_\alpha(x^a)c_\alpha(x) - \hat{u}^a \right]^2 \tag{64}$$

where N is the total number of nodal points x^a $(a = 1, 2, ..., N)$, $H(z)$ is the Heaviside unite step function

$$H(z) = \begin{cases} 0, & z \le 0 \\ 1, & z > 0 \end{cases}$$

and $\hat{u}^a$ are up to now unspecified coefficients, which will be determined by the physics of the problem. Recall that summation with respect to the repeated Greek subscripts is assumed in the range from 1 to μ.

Since $c_\alpha(x)$ should minimize the functional $J(x)$, we may write

$$\frac{\partial J(x)}{\partial c_\alpha(x)} = 0 = 2 \sum_{a=1}^{N} H\left(w^a(x)\right)w^a(x)p_\alpha(x^a)\left[p_\beta(x^a)c_\beta(x) - \hat{u}^a \right] \tag{65}$$

hence, we obtain the system of equations

$$A_{\alpha\beta}(x)c_\beta(x) = \sum_{a=1}^{N} B_{\alpha a}(x)\hat{u}^a \tag{66}$$

in which

$$A_{\alpha\beta}(x) = \sum_{a=1}^{N} H\left(w^a(x)\right)w^a(x)p_\alpha(x^a)p_\beta(x^a)$$

$$B_{\alpha a}(x) = H\left(w^a(x)\right)w^a(x)p_\alpha(x^a) \tag{67}$$

Having solved the system of equations (66) for $c_\alpha(x)$, we obtain

$$c_\alpha(x) = \sum_{a=1}^{N} A_{\alpha\gamma}^{-1}(x) B_{\gamma a}(x) \hat{u}^a \tag{68}$$

and finally, from (63), we have

$$u^h(x) = \sum_{a=1}^{N} H\left(w^a(x)\right) \phi^a(x) \hat{u}^a \tag{69}$$

where $\phi^a(x)$ are certain shape functions and $\hat{u}^a$ nodal values (or expansion coefficients) with

$$\phi^a(x) = p_\alpha(x) A_{\alpha\gamma}^{-1}(x) B_{\gamma a}(x) \tag{70}$$

Recall that $\hat{u}^a$ are fictitious nodal values, since $\hat{u}^a \neq u(x^a)$ and $\hat{u}^a \neq u^h(x^a)$, because

$$\phi^a(x^b) \neq \delta_{ab}.$$

Since the derived approximation is a domain-type approximation, we can get the approximation of the potential gradients by differentiating the approximation of the potential. Thus,

$$u_{,k}^h(x) = \sum_{a=1}^{N} H\left(w^a(x)\right) \phi_{,k}^a(x) \hat{u}^a \tag{71}$$

where the gradients of the shape functions are given as

$$\phi_{,k}^a(x) = p_{\alpha,k}(x) A_{\alpha\gamma}^{-1}(x) B_{\gamma a}(x) + p_\alpha(x) \left[A_{\alpha\gamma,k}^{-1}(x) B_{\gamma a}(x) + A_{\alpha\gamma}^{-1}(x) B_{\gamma a,k}(x) \right] \tag{72}$$

with

$$A_{\alpha\gamma,k}^{-1}(x) = -A_{\alpha\beta}^{-1}(x) A_{\beta\lambda,k}(x) A_{\lambda\gamma}^{-1}(x) \tag{73}$$

In a similar way, one could get also approximations of higher order derivatives of the primary field.

It is well known that the physics of any b.v.p. is determined by the boundary conditions and governing equations. That is why it is important to put the nodal points on the boundary of the analyzed domain as well as in its interior, when a domain-type approximation is utilized. Then, the system of algebraic equations for computation of the fictitious nodal values $\hat{u}^a$ can be constructed by

(i) collocation of the prescribed boundary conditions

$$u^h(\zeta^b) = \sum_{a=1}^{N} H\left(w^a(\zeta^b)\right) \phi^a(\zeta^b) \hat{u}^a = \overline{u}(\zeta^b) \ , \text{ if } \quad \zeta^b \in \partial\Omega_D \tag{74}$$

$$n_i(\zeta^b)u_{,i}^h(\zeta^b) = n_i(\zeta^b)\sum_{a=1}^{N} H\left(w^a(\zeta^b)\right)\phi_{,i}^a(\zeta^b)\hat{u}^a = \frac{\overline{q}(\zeta^b)}{k(\zeta^b)} \ ,$$

$$\text{if } \zeta^b \in (\partial\Omega_N - \partial\Omega_D) \qquad (75)$$

(ii) consideration of the local integral representations of the potential field at the interior nodal points x^c

$$\sum_{a=1}^{N}\hat{u}^a\left\{H\left(w^a(x^c)\right)\phi^a(x^c) - \int_{\Omega^c} H\left(w^a(x)\right)\phi_{,j}^a(x)\frac{k_{,j}(x)}{k(x)}G(x-x^c)d\Omega(x) + \right.$$

$$\left. + \int_{\partial\Omega^c} H\left(w^a(\eta)\right)\left[\phi^a(\eta)G_{,i}(\eta-x^c) - \phi_{,i}^a(\eta)G(\eta-x^c)\right]n_i(\eta)d\Gamma(\eta)\right\} = F^c \quad (76)$$

where F^c is defined after eq. (50). Basically, there is no restriction to the selection of the subdomain $\Omega^c \ni x^c$. Nevertheless, it is appropriate to consider sufficiently small subdomains, in order to receive sparse system matrix. Moreover, an appropriate arrangement of the shape of Ω^c can simplify the LIE. If Ω^c is a circle (or ball) with the radius r_o and centered at x^c, then the fundamental solution

$$G^*(r) = \begin{cases} -\dfrac{1}{2\pi}\ln\left(\dfrac{r}{r_o}\right) , & \text{for } 2-d \text{ problems} \\[4mm] \dfrac{1}{4\pi}\left(\dfrac{1}{r} - \dfrac{1}{r_o}\right) , & \text{for } 3-d \text{ problems} \end{cases} \qquad (77)$$

vanishes on the boundary $\partial\Omega^c$. Then, the LIE (76) becomes

$$\sum_{a=1}^{N}\hat{u}^a\left\{H\left(w^a(x^c)\right)\phi^a(x^c) - \int_{\Omega^c} H\left(w^a(x)\right)\phi_{,j}^a(x)\frac{k_{,j}(x)}{k(x)}G(x-x^c)d\Omega(x) + \right.$$

$$\left. + \int_{\partial\Omega^c} H\left(w^a(\eta)\right)\phi^a(\eta)G_{,i}(\eta-x^c)n_i(\eta)d\Gamma(\eta)\right\} = F^c \qquad (78)$$

Recall that the nodal values $\hat{u}^a$ have not a physical meaning. Having solved the system of algebraic equations for the fictitious nodal unknowns, one can reconstruct the physical solution by using eq. (69).

The integrations are performed in the global Cartesian coordinate system and the shape functions as well as their gradients are generated directly at the integration points numerically. For more details of the construction of the shape functions we refer the reader to [9, 10].

In contrast to the previous formulations based on the LBIE and MLS-approximation [10, 15], we propose to collocate directly the approximations for physical boundary quantities at nodal points on the boundary with taking into account the prescribed boundary conditions. Consequently, the formulation is really mesh-free because no integral equations are considered around the boundary nodal points and hence, it is unnecessary to model the boundary geometry for the integration over boundary segments. Moreover, the problem of singular boundary integrals is avoided, since the collocation point x^c is an interior point of the subdomain Ω^c ($x^c \notin \partial\Omega^c$).

4. Conclusions

The paper presents a new formulation for solution of boundary value problems in the potential theory. This formulation consists in:

(i) the domain-type approximation of the potential field

(ii) satisfaction of the prescribed boundary conditions at boundary nodes by collocating the approximations of the relevant quantities (derived from the approximation of the primary field – potential) at the boundary nodes

(iii) collocation of the local integral representations of the potential field at interior nodes.

Two kinds of the domain-type approximations are discussed with using either the discretization of the analyzed domain into standard finite elements or the MLS-approximation when a mesh-free scattering of nodal points is sufficient. In the first approach, certain grouping of nodal points into elements is required and the shape functions are known a priori in a closed form, while in the second approach no elements are required but the shape functions are more complex and evaluated only numerically at each integration point. Thus, the use of standard finite elements seems to be more effective in problems when re-meshing is not needed contrary to the mesh-free approach.

The proposed formulation possess certain advantages as compared with the standard boundary integral equations (considered on the global boundary of the analyzed domain) when the pure boundary formulation is not available. Such a case occurs if either the fundamental solution of the governing equation is not available or we do not use the original fundamental solution because of the enormous complexity of further calculations. In the proposed approach, the only unknowns are the nodal values of the primary field in contrast to the boundary-domain formulation, where the unknowns of the independent boundary densities are to be computed by solving the singular global BIE together with computation of unknowns at interior nodes. The LIE are non-singular and no BIE are considered at boundary nodes in this new approach.

Bearing in mind the necessity of regularization of standard BEM formulations, one can find sometime the proposed formulation more convenient even in case of problems when the pure boundary formulation is available, especially if we need to know the solution throughout the analyzed domain.

For simplicity, we explained the new formulation on potential problems with dealing some details only in two dimensions. Of course, the extension to solution of boundary value problems for arbitrary partial differential equations is possible.

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