

Boundary Element Methods in Solid Mechanics - a Tribute to Frank Rizzo

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Abstract: Frank Rizzo proposed the first Boundary Integral Equation (BIE) formulation for linear elasticity in a seminal paper in 1967. I have had the very good fortune to call Frank a friend of mine for many years. The present paper describes some of my experiences with the BIE over nearly thirty years, some of it in the context of Frank's pioneering contributions to the subject and my interactions with him.

The Early Years - 1970s: Like many important events in life, the start of my life-long love affair with Boundary Integral Equations (BIE) was somewhat accidental. It was the fall of 1974 and I was a bright-eyed and bushy-tailed first semester Assistant Professor at Cornell University. I had been advised to look for a line of inquiry in my research that was somewhat different from what I had been doing as a graduate student and as a Research Associate with my advisor Professor E.H. Lee at Stanford. To try to prove, as it were, that I had some ideas that were different from those of my illustrious advisor. It was now 1975 and the department Chair and my mentor, Professor Pao, had a suggestion for me. Would I like to attend a mini-symposium at RPI, on the Boundary Integral Equation Method, being organized by Cruse and Rizzo as part of an ASME meeting [5] ? Perhaps I would learn something about this new method that would be useful to my research. I was eager to go, but, it so happened that I had my eyes set on another meeting in San Francisco at exactly the same time. Among other things, I wanted to see my friends in Palo Alto whom I missed very much ! So, as a compromise (to, among other things, keep Professor Pao happy), I decided to send my graduate student, Virendra Kumar, to RPI, to learn about the new method, while I went on the longer plane ride to the west coast. Fortunately for me, Viren is a brilliant person (he has subsequently had a wonderful career at GE) and came back from RPI full of ideas about the BIE and the possibility of applying the method to elasto-viscoplastic problems that he and I were working on at that time. This meant, of course, that we should immediately start reading Rizzo's landmark paper (remarkably, a product of Frank's Ph.D. dissertation) on the BIE for 2-D linear elasticity [23], to be followed by Cruse's elegant generalization to 3-D elasticity [4]. (Some years later, Frank told me that people had expressed doubts on his choice of linear elasticity as his Ph.D. dissertation topic because they felt that there was nothing new to be discovered about this "old" topic. It was, after all, already the 1960s! Fortunately, for me in particular, and for the BIE community in general, Frank did not take this advice). Two years later, I published my first paper on the BIE [12], and I was passionately in love with the method.

Myself, Kumar, and later Mahesh Morjaria (also a brilliant person with a wonderful subsequent career at GE) continued our work on applications of the BIE in elasto-viscoplasticity and later on in inelastic fracture mechanics. Mahesh and I attended my first BIE meeting in Montreal in 1980 [25]. This was a wonderful meeting. Here I met several of my fellow BIE researchers who later became my life-long friends. It was a

real thrill to meet Frank Rizzo for the first time. (I heard a story that a young student, upon meeting Dr. Milne-Thompson for the first time, remarked in great surprise "Oh, Sir, I thought you were two !" At least I already knew that Frank Rizzo was one person !). A wonderful foundation to a great friendship was laid down at Montreal. Frank, as a researcher, and as a person, has been an inspiration to me ever since. The culmination of my efforts on the BIE during these early years was my first book on the subject [13]. (Looking back after all these years, I feel that perhaps I rushed the book a little bit. Maybe I should have taken more time to explain things a little better. Ah - well, to be young and foolish!)

The Middle Years - 1980s: The new decade brought new challenges. A hot topic in computational mechanics was the modeling of manufacturing processes such as metal forming. The Finite Element Method (FEM) was already making significant headway with this class of problems. This meant that one needed a BIE formulation for large strain - large deformation problems in elasto-plasticity and elasto-viscoplasticity. Abhijit Chandra and I embarked on this adventure and, again, thanks to Abhijit being an absolutely outstanding researcher, succeeded in coming up with such a formulation [1]. (Abhijit is now a very successful Chair Professor at Iowa State and was a colleague of Frank Rizzo for some years.) Inspired by the late Dr. Owen Richmond (at US Steel and later at Alcoa), I got interested in optimization and inverse problems around that time. I was fortunate to work on this class of problems with two absolutely outstanding students, Nicholas Zabararas and Qing Zhang. (Nicholas is now a very successful professor at Cornell and a world authority on inverse problems, and Qing is a researcher at Shell). Nicholas, I and Owen applied the BIE to solve an inverse heat transfer problem [28] while Qing, Abhijit and myself determined shape design sensitivities for nonlinear problems in solid mechanics using the BIE approach [29]. A book by Abhijit and myself [2] summarizes much of my BIE research during the 1980s.

I was very fortunate to interact with Frank Rizzo during this period. Frank visited Cornell to present a seminar and I visited him at Kentucky for the same purpose. I clearly remember a very stimulating discussion on the BIE that Frank and I had at Kentucky. This was followed by a memorable dinner in my honor at Frank's home which was attended by several of his very interesting Kentucky colleagues. I will always remember the warm hospitality of Frank and his charming wife Mary Lou during this visit. Then, in 1986, Frank, Dave Shippy and myself, were invited by Professor Pao to teach a short course on the BIE in Taipei, Taiwan. This trip was a real treat for me. The hospitality was superb - we were treated like royalty during this visit. I was also able, for the first time, to attend several expository lectures on the subject by a master - Frank Rizzo. The discussions that followed the lectures were also very stimulating.

The Recent Years - 1990s: I continued with BIE applications in nonlinear solid mechanics during the first half of this decade. A particularly productive period was my association with yet another fantastic student, Liang-Jenq Leu, currently a Professor at the National Taiwan University in Taipei, Taiwan. I am rather proud of some work we did together on shape optimization in elasticity and elasto-viscoplasticity [27]. I had the good fortune of visiting another wonderful BIE colleague - Marc Bonnet, at the Ecole

Polytechnique near Paris, France, on a sabbatical in 1994. The best part of this visit was to make another wonderful friend for life. I learnt a lot from Marc on Galerkin formulations and sensitivity analysis of linear problems. We worked together on a nonlinear problem - the implementation of a Consistent Tangent Operator based implicit BEM scheme for elastoplasticity. Several papers (e.g. [21] that involved yet another excellent graduate student - Harrison Poon, currently at HKS) resulted from this work. I had been thinking about the CTO for a while and was really happy to finally crack the problem with several key ideas coming from Marc.

A turning point in my research interests with the BIE came in the second half of this decade. For the first time in my career, I became interested in *linear* problems. My work during this time has included the Boundary Contour Method, the mesh-free Boundary Node Method (first proposed by myself and my wife Yu [14]) and error analysis and adaptivity (e.g. [3] carried out with two wonderfully talented graduate students Glaucio Paulino and Mandar Chati. Galucio, now an Associate Professor at UIUC, is a world leader on Functionally Graded Materials while Mandar, who came to Cornell via Iowa State upon the recommendation of Frank Rizzo, is doing very well at GE). My work during this period is being summarized in a book [20] that I and my wife are currently working on. (My wife did her Ph.D. in the area of the FEM ; some of our friends thought our marriage should be celebrated as a marriage of the BEM with the FEM !)

My recent work has finally come to meaningfully intersect that of Frank Rizzo on the subject of hypersingular integrals. I was very happy to extend ideas for regularization of hypersingular integrals, proposed by Rudolph [24] for potential theory, Rizzo et al. (e.g. [9] for acoustic and elastic wave scattering) and Cruse and Richardson [6] for elasticity - to problems of thermoelastic fracture mechanics [16] and elasto-plasticity [22]. But the work I am most proud of in this area is on an interpretation of the finite part of singular and hypersingular integrals of interest in the BIE [18]. I will devote the next section of this paper to a brief discussion of this idea and how it connects the work of several of my BIE colleagues, all of whom I admire greatly.

Finite Parts of Singular and Hypersingular Integrals: Mukherjee [18] presents a unified, consistent and practical definition of the finite part (FP) of a singular or hypersingular integral that is valid for both a regular as well as an irregular boundary collocation point.

Definition: Consider, for specificity, the space \mathbf{R}^3 , and let S be a surface in \mathbf{R}^3 . Let the points $\mathbf{x} \in S$ and $\boldsymbol{\xi} \notin S$. Also, let \hat{S} and $\bar{S} \subset \hat{S}$ be two neighborhoods (in S) of \mathbf{x} such that $\mathbf{x} \in \bar{S}$ (Figure 1). The point \mathbf{x} can be an irregular point on S .

Let the function $K(\mathbf{x}, \mathbf{y})$, $\mathbf{y} \in S$, have its only singularity at $\mathbf{x} = \mathbf{y}$ of the form $1/r^3$ where $r = |\mathbf{x} - \mathbf{y}|$, and let $\phi(\mathbf{y})$ be a function that has no singularity in S and is of class $C^{1,\alpha}$ at $\mathbf{y} = \mathbf{x}$ for some $\alpha > 0$.

The finite part of the integral:

$$I(\mathbf{x}) = \int_S K(\mathbf{x}, \mathbf{y})\phi(\mathbf{y})dS(\mathbf{y}) \quad (1)$$

is defined as:

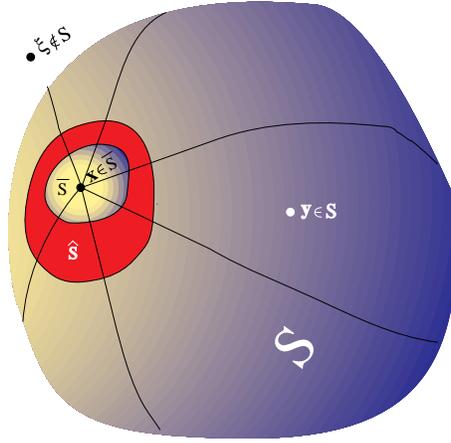


Figure 1: A surface S with regions \hat{S} and \bar{S} and points ξ , \mathbf{x} and \mathbf{y} (from [18])

$$\begin{aligned} \oint_S K(\mathbf{x}, \mathbf{y})\phi(\mathbf{y})dS(\mathbf{y}) &= \int_{S \setminus \hat{S}} K(\mathbf{x}, \mathbf{y})\phi(\mathbf{y})dS(\mathbf{y}) \\ &+ \int_{\hat{S}} K(\mathbf{x}, \mathbf{y})[\phi(\mathbf{y}) - \phi(\mathbf{x}) - \phi_{,p}(\mathbf{x})(y_p - x_p)]dS(\mathbf{y}) \\ &+ \phi(\mathbf{x})A(\hat{S}) + \phi_{,p}(\mathbf{x})B_p(\hat{S}) \end{aligned} \quad (2)$$

where \hat{S} is any arbitrary neighborhood (in S) of \mathbf{x} and:

$$A(\hat{S}) = \oint_{\hat{S}} K(\mathbf{x}, \mathbf{y})dS(\mathbf{y}) \quad (3)$$

$$B_p(\hat{S}) = \oint_{\hat{S}} K(\mathbf{x}, \mathbf{y})(y_p - x_p)dS(\mathbf{y}) \quad (4)$$

The above FP definition can be easily extended to any number of physical dimensions and any order of singularity of the kernel function $K(\mathbf{x}, \mathbf{y})$. Please refer to Toh and Mukherjee [26] for further discussion of a previous closely related FP definition for the case when \mathbf{x} is a regular point on S , and to Mukherjee [17] for a discussion of the relationship of this FP to the Cauchy Principal Value (CPV) of an integral when the CPV exists.

Evaluation of A and B : There are several equivalent ways for evaluating A and B .

Method one: Replace S by \hat{S} and \bar{S} by \bar{S} in equation (2). Now, setting $\phi(\mathbf{y}) = 1$ in (2) and using (3), one gets:

$$A(\hat{S}) - A(\bar{S}) = \int_{\hat{S} \setminus \bar{S}} K(\mathbf{x}, \mathbf{y})dS(\mathbf{y}) \quad (5)$$

Next, setting $\phi(\mathbf{y}) = (y_p - x_p)$ (note that, in this case, $\phi(\mathbf{x}) = 0$ and $\phi_{,p}(\mathbf{x}) = 1$) in (2), and using (4), one gets:

$$B_p(\hat{S}) - B_p(\bar{S}) = \int_{\hat{S} \setminus \bar{S}} K(\mathbf{x}, \mathbf{y})(y_p - x_p) dS(\mathbf{y}) \quad (6)$$

The formulae (5) and (6) are most useful for obtaining A and \mathbf{B} when \hat{S} is an open surface and Stoke regularization is employed. An example is the application of the FP definition (2) (for a regular collocation point) in Toh and Mukherjee [26], to regularize a hypersingular integral that appears in the HBIE formulation for the scattering of acoustic waves by a thin scatterer. The resulting regularized equation is shown in [26] to be equivalent to the result of Krishnasamy et al. [9]. Equations (5) and (6) are also used in Mukherjee and Mukherjee [15].

Method two: From equation (5):

$$A(\hat{S}) - A(\bar{S}) = \int_{\hat{S} \setminus \bar{S}} K(\mathbf{x}, \mathbf{y}) dS(\mathbf{y}) = \lim_{\xi \rightarrow \mathbf{x}} \int_{\hat{S} \setminus \bar{S}} K(\xi, \mathbf{y}) dS(\mathbf{y}) \quad (7)$$

The second equality above holds since $K(\mathbf{x}, \mathbf{y})$ is regular for $\mathbf{x} \in \bar{S}$ and $\mathbf{y} \in \hat{S} \setminus \bar{S}$. Assuming that the limits $\lim_{\xi \rightarrow \mathbf{x}} \int_{\hat{S}} K(\xi, \mathbf{y}) dS(\mathbf{y})$ and $\lim_{\xi \rightarrow \mathbf{x}} \int_{\bar{S}} K(\xi, \mathbf{y}) dS(\mathbf{y})$ exist, then:

$$A(\hat{S}) = \lim_{\xi \rightarrow \mathbf{x}} \int_{\hat{S}} K(\xi, \mathbf{y}) dS(\mathbf{y}) \quad (8)$$

Similarly:

$$B_p(\hat{S}) = \lim_{\xi \rightarrow \mathbf{x}} \int_{\hat{S}} K(\xi, \mathbf{y})(y_p - x_p) dS(\mathbf{y}) \quad (9)$$

Equations (8) and (9) are most useful for evaluating A and \mathbf{B} when $\hat{S} = \partial B$, a closed surface that is the entire boundary of a body B .

Method three: A third way for evaluation of A and \mathbf{B} is to use an auxiliary surface (or ‘‘tent’’) as first proposed for fracture mechanics analysis by Lutz et al. [10] (see, also, Mukherjee et al. [16] and Mukherjee [19]). This method is useful if S is an open surface.

The FP and the LTB: There is a very simple connection between the FP, defined above, and the Limit to the Boundary (LTB) approach employed by Gray and his co-authors. With, as before, $\xi \notin S$, $\mathbf{x} \in S$ (\mathbf{x} can be an irregular point on S), $K(\mathbf{x}, \mathbf{y}) = \mathcal{O}(|\mathbf{x} - \mathbf{y}|^{-3})$ as $\mathbf{y} \rightarrow \mathbf{x}$ and $\phi(\mathbf{y}) \in C^{1,\alpha}$ at $\mathbf{y} = \mathbf{x}$, this can be stated as:

$$\lim_{\xi \rightarrow \mathbf{x}} \int_S K(\xi, \mathbf{y}) \phi(\mathbf{y}) dS(\mathbf{y}) = \int_S K(\mathbf{x}, \mathbf{y}) \phi(\mathbf{y}) dS(\mathbf{y}) \quad (10)$$

Of course, ξ can approach \mathbf{x} from either side of S .

Proof of equation (10): Consider the first and second terms on the right hand side of equation (2). Since these integrands are regular in their respective domains of integration, one has:

$$\int_{S \setminus \hat{S}} K(\mathbf{x}, \mathbf{y}) \phi(\mathbf{y}) dS(\mathbf{y}) = \lim_{\boldsymbol{\xi} \rightarrow \mathbf{x}} \int_{S \setminus \hat{S}} K(\boldsymbol{\xi}, \mathbf{y}) \phi(\mathbf{y}) dS(\mathbf{y}) \quad (11)$$

and

$$\begin{aligned} & \int_{\hat{S}} K(\mathbf{x}, \mathbf{y}) [\phi(\mathbf{y}) - \phi(\mathbf{x}) - \phi_{,p}(\mathbf{x})(y_p - x_p)] dS(\mathbf{y}) \\ &= \lim_{\boldsymbol{\xi} \rightarrow \mathbf{x}} \int_{\hat{S}} K(\boldsymbol{\xi}, \mathbf{y}) [\phi(\mathbf{y}) - \phi(\boldsymbol{\xi}) - \phi_{,p}(\boldsymbol{\xi})(y_p - \xi_p)] dS(\mathbf{y}) \end{aligned} \quad (12)$$

Use of equations (8, 9, 11 and 12) in (2) proves equation (10).

Connections with Related Work: Interesting connections of the above FP definition, with related work by several other BIE researchers, are proved in [18]. With a vanishing exclusion zone, one recovers, from the proposed FP definition (equation (2)), the results in Guiggiani [8], Guiggiani et al. [7] and Mantić and Paris [11], while, with a complete exclusion zone, one recovers the results from Rudolphi [24]. Finally, the regularized stress BIE in Cruse and Richardson [6] can also be obtained from (2) with a complete exclusion zone.

The Future: Where do we go from here with the BIE ? I feel that the BIE is a niche method, and, as such, it is not a good idea to compete directly with the much more versatile FEM. Thus, problems with several coupled fields, or with extensive nonlinearities, are perhaps best left in the realm of the FEM. Below is a (partial) list of problems that, I feel, offer advantages to the BIE compared to other numerical methods.

- BIE and mesh-free BIE for large linear problems in complex domains
- BIE accelerated with Fast Multipole (FM) type methods
- infinite domain problems (e.g. in acoustics)
- problems involving very thin objects or features (e.g. thin shells, fracture mechanics)
- error analysis and adaptivity
- sensitivity analysis, optimization and inverse problems
- problems with many microscopic interacting cracks, voids and other features
- nonlinear problems with limited regions of plasticity
- coupling of the BIE with other computational techniques for appropriate problems

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