Early Development of the BEM at the University of Kentucky

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Introduction

I am very pleased to have this opportunity to honor Frank J. Rizzo, following his retirement, by contributing to this volume. My professional and close personal association with Frank for two decades was one of the great privileges and pleasures of my life. Our lives were closely entwined not only as teaching and research colleagues but also as good friends.

During the course of our work together I developed an immense respect for Frank's vision, his creativity, his care in being clear, complete and accurate, and his understanding of the foundations of solid mechanics. In addition, he was adept in communicating with others, such as colleagues and students, had a great capacity for hard work, and showed a remarkable willingness to plow through some of the less pleasant aspects of research, such as writing proposals and reports.

Frank has referred briefly to our collaboration on the development of the Boundary Element Method (BEM) in a 1989 review paper [1]. My purpose in this paper is to elaborate somewhat on that portion of his paper. (Because my recollection of this history, which began thirty-five years ago, is somewhat hazy, Frank will no doubt recognize some inaccuracies in my account below.)

When Frank became a faculty member at the University of Kentucky in 1966 he had already written his seminal paper on the use of integral equations to solve elasticity problems [2]. Before long, Frank discovered that I was very much interested in and involved with the use of computers. He approached me, described his research interest, and proposed that we collaborate on future research of this kind. This appeared to me to have the potential for some interesting computer applications, so I agreed to work with him. At that time we never imagined that this collaboration would have as great an impact on our professional careers as it turned out to have.

I gratefully acknowledge the enhancement that our collaboration received from the able assistance of several very bright and talented graduate students. These students are identified as co-authors of various papers as given below in the references of this paper.

In the Beginning

With no computer code in hand, Frank and I proceeded to develop from scratch some ad hoc (for specific geometries) direct boundary integral equation code for solution of plane elastostatics problems. Frank laid out the mathematical formulation, which assumed piecewise-constant boundary variables, while I looked over his shoulder and put in my two-cents worth. (At this point, we made all approximations analytically, so the numerical solution employed exact methods.) Because I enjoyed the computer work much more than Frank did, I did most of the actual coding, but Frank likewise looked over **my** shoulder and made many helpful suggestions. (I recall his being amused at times at the whimsical variable names which I conjured up.) Our first (test) code was for the solution of a very simple problem. We spent at least one Saturday at the campus computing center agonizing over that code before shouting "Eureka!". (Back in those days we entered programs and data into the mainframe computer with punched cards and batch processing--no PC's or even terminals.)

Having that small success behind us, we were poised to apply the boundary integral equation (BIE) method to more difficult problems. At this point we could have chosen any of many possible applications, because we were in unexplored territory. Ironically, reviewers of some of our early manuscripts and research proposals complained that our work or proposed work was too speculative, whereas after the method had become somewhat established just a few years later, some reviewers complained that our work (of the same type as earlier) involved just another mundane application.

We chose to proceed by writing new, still ad hoc, code for solving some elasticity problems involving inclusions [3]. Initially, we made the erroneous assumption that the mismatch between an elliptical inclusion and the corresponding opening in which it fit was uniform. This led to some very weird results. Eventually, we realized that the mismatch at each point on the edge of the inclusion must be proportional to the coordinates of the ellipse, with origin at its center.

Our next foray was application of the method to analysis of plane anisotropic elastic bodies [4]. This involved more complicated kernel functions, which, in turn, required new consideration of existence and uniqueness of solutions.

The Time Dimension

We then decided that it would be exciting to add a new dimension (time) to our work by investigating some nonstatic phenomena, starting with transient heat conduction [5]. Here, of course, we were dealing only with a scalar field variable, temperature, in contrast to the vector field variables of elasticity. We chose to develop a BIE formulation for the Laplace-transform space. Solutions in the time domain (with temperature changing monotonically with time) were obtained by numerical transform inversion of results from the transform space obtained for several values of the transform parameter.

We noted that the field variables of quasi-static viscoelasticity have time-variations of the same character as temperature in transient heat conduction. Consequently, a natural extension of our work with transient heat conduction was the development of a BIE formulation for quasi-static viscoelasticity in the Laplace-transform space [6]. This was facilitated by use of the correspondence principle. It represented a return to the complexity of vector field variables, but now with the additional time dimension involved.

In our example problems of transient heat conduction and quasi-static viscoelasticity, the field variables had been monotonic functions of time. The method we had used for numerical transform inversion in those problems had been designed for such functions. However, we wondered whether that method could be used also for oscillatory functions. So we did some unpublished work with a BIE formulation for linear viscoelastic wave propagation in the Laplace-transform space. The inversion method provided very good results for such problems involving functions with a little oscillation.

New Techniques

Up to this point in our research we continued to use ad hoc computer codes (for two spatial dimensions and specific geometries), which were based on the assumption of piecewise-constant boundary variables, analytical approximations, and exact methods for numerical solution. Then we became aware of the work of Lachat and Watson ([7], [8]), who employed in their boundary-integral procedures some mathematical techniques that were widely used in finite-element analysis: the use of shape functions to approximate boundary geometry and boundary variables (e.g., displacement and traction); and the use of approximate quadrature formulas (e.g., Gaussian quadrature). We realized that these techniques represented an important advancement in boundary integral methods and decided to adopt them for our work. In all subsequent work we utilized isoparametric quadratic boundary elements. (We were chagrined not to have been the first to adopt these techniques.)

We then set about to develop general-purpose solution procedures and computer codes (for two and three spatial dimensions) which utilized these techniques (with quadratic shape functions) and were capable of handling arbitrary geometries and boundary conditions. Initially, these were for problems associated with the Poisson equation (e.g., steady-state heat conduction) and isotropic thermoelastostatics.

One of the first applications which we made of our new capability was to show its utility for solving two-dimensional electromagnetic field problems associated with the Poisson equation [9].

Then we demonstrated that we could readily solve three-dimensional elasticity problems involving stress concentrations and/or thermal stresses [10].

At about this time we were awarded a contract by the Air Force Office of Scientific Research for the study of thermoelastic stress concentrations adjacent to cooling holes in jet-engine turbine blades. The results of this work were recorded in several progress reports and a chapter of a book [11].

Singular Integrands

As is well known, boundary integral equations involve some integrals with singular integrands. Those integrals which must be evaluated by approximate quadrature methods are of order log(r) for two-dimensional problems and order 1/r for three-dimensional problems. Our approach to the evaluation of such integrals was to eliminate the singularities by appropriate changes of integration variables. For two-dimensional problems, the use of shape functions results in the variable of integration being the intrinsic coordinate ξ. By the introduction of a variable y such that $\xi = y^2$, the new integrand in terms of new integration variable y is nonsingular. Similarly, for three-dimensional problems, the use of shape functions results in the two variables of integration being the intrinsic coordinates ξ_1 and ξ_2, which are cartesian coordinates in the intrinsic-coordinate space. We introduced polar coordinates ρ and ϕ, such that $\xi_1 = \rho * \cos(\phi)$ and $\xi_2 = \rho * \sin(\phi)$. Again, the new integrand in terms of new integration variables ρ and ϕ is nonsingular. Our mathematician colleague Graeme Fairweather was horrified when he learned how we handled the singular-integrand situations. As a numerical analyst his inclination was instead to use weighted quadrature formulas. So in collaboration with Frank and myself, Graeme studied the use of various approaches to handling the singular integrands for three-dimensional problems [12]. The conclusion was that our change-of-variable approach was the most effective of the various approaches studied.

Other Problem Types

Our attention next was drawn to the use of boundary-integral methods for the solution of boundary-value problems involving bodies of axisymmetric geometry. Noting that BIE formulations for solving elasticity problems involving axisymmetric bodies with axisymmetric loading had appeared in the literature, we set about to develop a formulation that would handle arbitrary (non-axisymmetric) loading of axisymmetric bodies [13] by means of Fourier series expansions, as had been done previously with finite-elements. In effect, the formulation reduces a three-dimensional problem to a sequence of one-dimensional problems. Subsequently we provided further details of the formulation and presented illustrative examples for potential problems [14] and elastostatics ([15], [16], [17]). We found that extreme care was required in the formulation because of the cylindrical coordinates used [18]. In particular, finding an obscure error, related to the coordinates, that we had made in accounting for the effects of body forces resulted in some time-consuming frustration.

A survey of the literature on wave propagation renewed our interest in time-varying phenomena. Consequently, we began an extensive research project to develop and utilize boundary-integral capabilities for solution of problems of wave radiation and scattering in fluids and elastic solids. Initially, we restricted our attention to steady-state problems, but eventually we considered transient problems, utilizing both Fourier-transform and Laplace-transform approaches. Our studies included consideration of axisymmetric bodies and infinite full- and half-spaces. An additional collaborator on portions of this research was our colleague Andy Seybert. This work was supported by a grant from the National Science Foundation and a contract with the U.S. Office of Naval Research and resulted in several publications ([19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32]) The final phase of our collaboration, begun before Frank left Kentucky, dealt with linear elastic fracture problems ([33], [34], [35]). Knowing the limiting variation of displacement and traction with position in the region of a crack tip, we utilized special shape functions having the known variation (singular for traction). As before, we used (different) changes of variables to eliminate singular integrands.

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