Springs, Formulas and Flatland: A Path to Boundary Integral Methods in Elasticity

A Memoir By

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Preface

This memoir is about some embryonic thoughts and experiences with what is now often called the *Direct Boundary Integral Method* for boundary value problems in 'classical elastostatics.' It covers a very early period for me, from about 1954 to 1965, when I wrestled with relevant theoretical and mathematical issues, which I thought were important, primarily in the last two or three of those years. This was before I became aware of other contributors to the method, who are not mentioned here, and before they became aware of my work. So this is not a memoir largely filled with recollections of people and places; although I believe that subsequent interactions with fellow contributors and users of computational methods have become the most important part of my research experience. In any case, I understand that memoirs and papers of a more technical nature, written by some of those contributors, are part of these EJBE volumes. Those writings have bearing, no doubt, on the present memoir and much more, over a long period of time, and I look forward to reading them all.

Springs

I don't remember exactly when I first encountered the equation $P = K\,\delta$ for a linear elastic spring, where P is the load, δ the spring deflection, and K the spring constant. It must have been in my first course of study in physics. This spring equation probably represented my first mathematical model. The ideas of *physical variables*, P and δ, as distinct from *parameters or constants* K which characterize different springs, first emerged then. I noticed that each quantity has different physical dimensions and learned too that one could conduct experiments, physical or just mental ones, based on such a model. If P is known, one can measure δ such that K for a given spring can be found. Then if the spring is to be stretched a specified amount δ, the equation gives the P required for the stretch. Or if a P is specified, the equation gives the associated δ.

Sometime later, I learned that for a prismatic linear elastic bar of length L and cross-sectional area A, undergoing simple tension or compression, an equation comparable to the one for the spring can be written as $P = (AE/L)\,\delta$, where E is Young's Modulus of the bar material, and P and δ are the end-load and length-change of the bar, respectively, as with the spring. Thus, one may define an 'equivalent spring constant' for the bar as $K \equiv AE/L$. The above observations about physical and mental experiments are a bit richer with the bar perhaps, but not essentially different. This was, in any case, 'my first elasticity problem,' albeit a *one dimensional* one, in which Young's Modulus was explicit. Yet this problem proved to be more important to me much later, than I could possibly have realized then.

Formulas

In college at Illinois, I began studying linear Elasticity Theory (ET) in earnest where little time if any was spent on one-dimensional problems. Our course moved quickly on to some general concepts and special methods of attack for certain elementary problems in both *two and three spatial dimensions*. Following this, in some more theoretical courses from Professor Marvin Stippes, we studied a variety of quite mathematical ideas in classical elasticity the most memorable now being *representation integrals* for elastic field variables in terms of specifiable boundary data and some fundamental tensor fields. Throughout, regardless of the complexity of many of the representations, or the simplicity of others, it seemed that *any elastic body* of arbitrary shape, under two- or three-dimensional assumptions, could be thought of as a *spring* in some general sense. Nevertheless, this observation seemed to be of little help in solving problems involving more complicated geometries than prismatic bars. Yet it contributed to a vague idea of a practically useful attack on such problems, perhaps with the aid of then-emerging computing facilities. Such an idea seemed very attractive to me.

One of the representation integrals we studied in Stippes' classes, for the displacement field *u(x)* at all points *x* throughout virtually any linearly-elastic body *B* with surface *S*, involves an integral, *over S alone*, of relatively few basic quantities. Specifically, this integral known as *Somigliana's Identity* (cf. [1]) has the pair *{t(y), u(y)}* in its integrand, where *y* is an arbitrary point on *S* and *t(y)* is the surface traction. These data on *S* pertain to one and the same equilibrated deformed state, characterized by *u(x)* for an arbitrary body *B*. The only other ingredients in the integrand are *known* two-point tensor quantities. These tensors, *T(x,y)* and *U(x,y),* are the *same for every problem, for any given B*, and they describe the stress and displacement fields, known as Kelvin's solution [2] at a point *y*, due to a point force acting at another point *x* in an infinite elastic space.

Somigliana's Identity is an elegant construct but since, loosely speaking, only 'half' of the pair *{t(y), u(y}* may be specified at the start of a well-posed boundary value problem, it is only an identity and *is not* an integral representation for *the solution u(x)* of any *specific* problem. **{F1}** Because of this shortcoming, various theoretical additions and modifications to the derivation process of Somigliana's Integral were offered historically for the purpose of obtaining an integral which is at least a *formal* representation for the *solution* to a specific problem, and *not merely an identity*. The objective was to eliminate, in the integrand of Somigliana's Integral, dependence on the unspecified 'half' of the pair of boundary data. Unfortunately, this objective, whenever realized, came at a great price. New functions which *depend on B* needed to be found and introduced into the integrands. These new functions, which are 2^{nd} order tensors for ET, also *depend on <u>which</u> half of the respective pairs* are specified.

Finding such functions forty years ago, in any kind of usable form, seemed, in general, an impractical route towards a goal I envisioned at the time. This goal was to find a systematic, unified solution process, via a single integral representation, if possible, for any type of well-posed problem in ET. This process, a numerical one in general, would exploit the ability of the then-new crop of digital computers to do arithmetic. But I thought that some way to avoid the mentioned region- and problem-dependent tensors was needed.

It seemed important, at this stage, to have a closer look at the pair $\{t(y), u(y)\}$ in light of the concept of a well-posed boundary value problem. At the same time, a rather simple, perhaps even trivial thought, which first surfaced during my earliest elasticity studies, kept coming back to me. That is, the equation $P = (AE/L)\ \delta$ for the bar-as-spring, gives the (relative) end *displacements* of the bar if the end loads, or *tractions*, are specified; and the *reverse* is true as well. Upon specification of either end point or 'boundary quantity' initially, this spring equation gives the other.

However, in a two- (2D) or three-dimensional (3D) problem, the situation is apparently more complicated. Either traction $t(y)$ or displacement $u(y)$, in the 'pair' $\{t(y), u(y)\}$ may be specified *at any point* y on the line *S*, or surface *S*, which bounds a body *B*, for a well-posed problem in 2D and 3D. These problems are called *traction* and *displacement* boundary value problems, respectively. *Mixed* problems, wherein $t(y)$ is prescribed over part of *S*, and $u(y)$ over the remaining part may also be well posed. **{F2}** In all three of these cases, the unprescribed member of the pair $\{t(y), u(y)\}$ is not so readily obtainable for any except perhaps the one-dimensional case.

Indeed, it appears in general, that it is impossible to find the unprescribed member of the pair (except, of course by physical measurement on *S*) without first finding the displacement $u(x)$ and stress $\sigma(x)$ for all *x* throughout *B*. For example, if one deforms a thin flat elastic plate of trapezoidal shape, with prescribed in-plane tractions $t(y)$, on say two of its four sides (and $t(y) = 0$ on the other two), the associated displacements $u(y)$ on all of the sides apparently remain unknown until a formula can be written for $u(y)$ based on knowledge of $u(x)$ throughout the plate *B*. Actually, even for the one-dimensional bar-as-spring, the 'field' $u(x) = (P/AE)\ x$ technically comes first (so does $\sigma(x) = P/A$ for all *x*) such that with $u(x) = 0$ and $u(L) = \delta$, we have the equation relating *P* and δ which I've been so fond of.

Why was I worried about such things? Why did the idea of finding the unprescribed part of the pair $\{t(y), u(y)\}$, short of solving the whole problem first, keep nagging me?

In our theoretical courses, Stippes showed us that ET could be viewed as a vector version of the more familiar scalar Potential Theory (PT) such as found in Kellogg's book [3]. With this view, The Navier Equations of elasticity are vector analogues of Laplace's Equation, and certain representation integrals in ET are analogues of the single- and double-layer potentials in PT. In particular, I was able to view Somigliana's Identity as the vector analogue of Green's (3rd) Identity for Laplace's

Equation. In Green's Identity, there appears the pair *{φ(y), φ'(y)}* consisting of the boundary values of a desired harmonic function *φ*, and its normal derivative *φ'*, on *S* of a region *B*, in which the harmonic field *φ(x)* is defined. These functions are the scalar analogues of the vectors *u(y)* and *t(y)*, respectively, in ET. Also in Green's Identity, there appear fundamental known scalar functions, *G(x,y)* and *N(x,y)*. These functions, similar to tensors *T(x,y)* and *U(x,y)* in Somigliana's identity, are the *same for every PT problem for any given B*.

At the best possible stage, after thinking about all of the above issues, there appeared a paper by Jaswon [4] wherein he suggested solving 2D problems in PT by means of an attack (numerically in general) on some singular integral equations. At this point, a 'light bulb' appeared in my consciousness - dim at first, but it grew in brightness with time.

Jaswon's integral equations were *boundary integral equations defined on (the line boundary) S alone*, derived from Green's Identity, by taking the limit as the field point *x* in *B* goes to a boundary point *z* on *S*. From this limit, with due regard to the continuity or lack of it, for each integral in the limit, a formula is obtained, albeit with undefined integrands (when *z* coincides with *y* on *S*). However since these undefined integrands are known to be integrable, Jaswon noticed that the resulting formula, sometimes called *Green's Boundary Formula*, is a vehicle by which *unprescribed parts of the pair {φ(y), φ'(y)} may be found*. The *solution* for *φ(x)* is then obtained from Green's Identity with *x* back in *B*, since at this stage both members of the pair *{φ(y), φ'(y)}*, for a given PT problem, are known. Green's Identity thereby becomes the solution to the originally-posed problem, satisfying Laplace's Equation at every *x* in *B* and meeting the required boundary conditions at all *y* on *S*.

After I had digested Jaswon's paper, the 'light bulb' brightened considerably. I dared to think of something which might bear the name *'Somigliana's Boundary Formula,'* analogous to that of Green, to be used for purposes, analogous to those of Jaswon's, but this time for all three types of boundary value problems in ET, rather than the comparatively simpler problems in PT. If this idea would work for ET, like it seemed to for PT, the usual strategy in ET, wherein field solutions *u(x)* and *σ(x)* are found first throughout B *and then u(y)* and *t(y)* on *S*, could be *reversed*. The idea of finding 'compatible' boundary data *{t(y), u(y)}* on *S*, i.e., data corresponding to one and the same equilibrated deformed state, from a single formula defined on *S* alone, for virtually every type of boundary value problem, regardless of spatial dimension, without explicit concern first for the fields *u(x)* and *σ(x)* throughout *B* (unless desired), seemed like an idea worth pursuing. What's more, inherent in the idea, the real work including all quadratures would be done *on the boundary S* of *B* which is, of course, a spatial region having one less dimension than *B* itself. That is, *S* is a surface for three-dimensional *B*, a line for two-dimensional *B*, and, as we have noted above, *S* consists of the two end points of the one-dimensional 'spring like' bar.

So pursue the idea I did. I engaged a student, C.C. Chang, for help with computer programming, and attempted for ET in 2D what Jaswon and his colleagues, Symm and Ponter [5] and [6], did for PT in 2D. To the surprise and, I'm happy to say, the occasional delight of all concerned, the attempt was successful.

The result [1] was published in April, 1967 and recognized in 1993 by the American Society of Mechanical Engineers, with their 'Worcester Reed Warner' Medal, following a most generous nomination and support from some colleagues. One such award is given annually by ASME for an 'Outstanding Contribution to the Permanent Literature of Engineering.' This recognition was rewarding for many reasons not the least of which was its bearing on the remark, made years earlier, by a member of my thesis committee. That remark was "…. this seems like nice work, Frank, but it's not really engineering…." to which, in all honesty at that time, I had to agree.

Reflections

Looking back, I believe that my seemingly naive 'obsession' with the bar-spring problem, together with the view that any elastic body *B*, could be regarded as a spring, regardless of dimension, were important preparation for me to think along the lines presented here. Essential were the elegant lectures by Stippes on Somigliana, with Stippes' view that ET was in essence PT in vector form. Essential too, of course, was Jaswon's bold analytical scheme of seeing Green's Boundary Formula as a constraint between $\varphi(y)$ and $\varphi'(y)$ corresponding to one and the same harmonic function in *B*, from which an unprescribed part of this pair could be determined. Finally, and very encouraging, Jaswon, Symm and Ponter verified that their ideas were viable numerically, although the numerical methods I chose, to evaluate improper integrals, differed considerably from theirs.

It is illuminating, to see how individual minds have interests and make contributions to a given enterprise consistent with and sometimes confined by those interests. Stippes, the consummate analyst, set the stage and provided the advanced analytical tools for me to think with, yet back then he seemed relatively disinterested in practical strategies, especially in numerical work, that might be connected with ET. That attitude changed, towards my research with the Somigliana formulas at least. He was away on sabbatical for the year most of my analytical work was done, and he was essentially unaware of what I had done. Stippes was my major professor, so I was pleased, to say the least, when *he* was pleased with what I showed him upon his return. For their part, Jaswon and his colleagues seemed interested in both analysis and computation, but whatever their interest in ET, in general, or knowledge of Somigliana, in particular, no evidence of the idea of ET as a vector PT was apparent in their work.

I remember thinking, long after our initial success, that Somigliana's work was done late in the 19^{th} Century – in 1885, I believe. His Integral Identity was developed obviously for a different purpose than the role it was playing for us. How nice, I thought, that an integral expression, of almost exclusively theoretical value, after so many years, might form the basis for a computer-based general solution method for practical problems in engineering and applied physics. It is *'old math in new garments.'* I wonder if Somigliana would have been pleased to see his work used this way.

The process described above has been called the *Direct Boundary Integral Method for Elasticity.* However, because of the popularity and importance of the method's challenging numerical implementation, and some similarities it shares with the powerful 'Finite Element Method,' the process usually falls under the name *Boundary Elements.* In any case, so many people in so many places over so many years now have been involved with ideas like those in this memoir that they have become useful, not only for potential theory and elasticity, but also for other mathematically-comparable disciplines in 2D and 3D, as is now well known, and is no doubt evident in some of the papers in these EJBE volumes.

A realization, which proved useful throughout my research, is that Gauss' Divergence Theorem for integral calculus in three dimensions is identical with Green's Theorem in two dimensions; and what was less obvious, and honestly a bit of a surprise to me at first, is the fact that in one dimension, both theorems reduce to the Fundamental Theorem of Integral Calculus. Part of the insight of Green and Gauss was, no doubt, to *seek* their 'Theorems' as extensions, to higher dimensions, of that fundamental theorem.

Green and Somigliana, went on to consider the divergence (or some generalized divergence) of the product of a desired potential or elastic field in B with a generalized gradient of a carefully selected special, function, i.e., the 'fundamental solution' to the governing field equation(s). Then, with the aid of some limits on the integral of that divergence, over a spatial or plane region B according to the Divergence Theorem, the desired field emerges. This is expressed in terms of an integral over S of B which contains only boundary pairs e.g., $\{u(y),\ t(y)\}$ and $U(x,y),\ T(x,y)$ (cf. [7]). Then a few more limits, as x goes to z on S, as argued previously, give rise to boundary integral solution strategies such as described above.

Having been involved so intimately with 'boundary research', by whatever name, has been an experience near the top of my list of life's wonderful experiences – exceeded perhaps only by the warmth of the relationships I've enjoyed with my family and friends, including the students and colleagues connected with the work for the last 40 years, all of whom I regard as friends, as well. I believe that one's view of the universe falls into only one of two camps; it is an *accident* or a *gift*. I see no other possibilities. My view falls in the latter camp, and I've been taught that the appropriate response to a gift is to say thanks. Thus I offer my most sincere thanks here: to the Creator of all that is, and to my parents, for the gift of life, and to you my friends who are part of that gift. Thanks too for perhaps participating in these EJBE Volumes, and for all of your individual contributions, both professional and personal (you know what they are), toward making life with boundary research so rich and meaningful.

Flatland

Thinking about bodies B with boundaries S, in one, two, and three dimensions, and indeed the whole boundary formula strategy just described, always reminds me of the book *Flatland* [8] written about 120 years ago by Edwin Abbott. This delightful fantasy -

at first glance a children's book, but written for all ages, was introduced to me by my dear friend and longtime colleague, David Shippy, more than thirty five years ago. If you haven't read *Flatland* allow me to recommend it. Having read the present article, I suspect you will see why I like *Flatland* so much. Regardless, share the book with a curious child or with an adult who "seems to know everything." You'll have some fun.

References

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FOOTNOTES

** {F1}** Knowledge of *u(x)* throughout *B* completely characterizes the equilibrated deformed state. Quantities of interest to engineers, such as stresses and strains throughout *B*, are readily obtained from first derivatives of *u*.

{F2} Other mixed problems include prescriptions of linear combinations of *t(y)* and *u(y)* over *S*, or selected corresponding components of *t(y) or u(y)* [but not both at the same *y*] over *S* or portions of *S* (cf. [1]). Body forces and thermal influences are assumed zero here et. seq.
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