A DRBEM For Time-Dependent Infiltration from Periodic Irrigation Channels in a Homogeneous Soil

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Abstract

In this paper, a problem involving time-dependent water flow in a homogeneous soil is considered. The problem involves water infiltration from periodic identical trapezoidal channels. A governing equation of the problem is the Richard's equation, which can be studied more conveniently by transforming the equation to a Helmholtz equation using the Kirchhoff transformation with dimensionless variables and a Laplace's transform. A dual reciprocity boundary element method (DRBEM) is employed to solve the Helmholtz equation numerically. Results obtained are found to be physically reasonable.

Keywords : Dual-reciprocity boundary element method, periodic identical irrigation channels, infiltration.

1 Introduction

Studies of water infiltration into soils have been carried out by a number of researchers. Waechter and Mandal studied steady infiltration from a semicircular cylindrical trench and hemispherical pond [17]. Steady infiltration from buried and surface cavities have been considered by Pullan and Collins [13]. Problems involving steady infiltration from irrigation channels have been investigated by Batu [5], Azis et al [4], Clements et al [8], Lobo et al [11], and Solekhudin and Ang [15, 16]. Time-dependent infiltration problems from a semi-circular channels have been examined by Clements et al [9].

In the current study, we investigate solutions to time-dependent infiltration from periodic channels. Trapezoidal channels are considered here, as these channels are commonly found in farms and plantations requiring furrow irrigation in developing countries. A set of transformations is employed to transform the governing equation for water flow in soil to a Helmholtz equation. A numerical scheme based on a dual reciprocity boundary element method, or DRBEM, is constructed using an integral formulation. The method is tested on a problem involving time-dependent infiltration from periodic trapezoidal channels. The solutions obtained are presented graphically.

2 Problem formulation

Using a coordinate system $OXYZ$ with OZ pointing positively downward, we consider homogeneous pima clay loam soil in the region $Z \geq 0$. Periodic trapezoidal channels are created on the surface of the soil. For every unit length in the OY direction, the channel has a sunken surface area of $2L$ square units. The distance between centres of two consecutive channels is $2(L + D)$. It is assumed that cross-sectional geometry of the channels does not vary in the OY direction and is symmetrical about the planes $X = \pm k(L + D)$, for $k = 0, \pm 1, \pm 2, \cdots$. The geometry of the channels is illustrated in Figure 1.

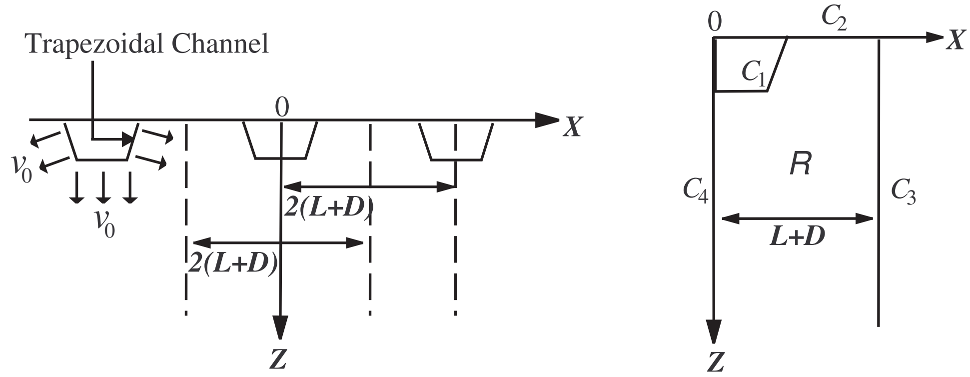

Figure 1: Geometry of periodic trapezoidal channels

Due to the symmetry of the problem, it is sufficient to consider the semi-infinite region bounded by $0 \leq X \leq L + D$ and $Z \geq 0$. This region is represented by R bounded by C_1, C_2, C_3, and C_4. C_1 is the boundary along the surface of the channel, and C_2 is the boundary along the surface of the soil outside the channel. The fluxes across C_1 and C_2 are v_0 and 0 respectively. C_3 and C_4 are the boundary along $X = L + D$ and $X = 0$, and they have zero fluxes.

3 Basic equations

Time-dependent infiltration is governed by the following Richard's equation

$$\frac{\partial \theta}{\partial T} = \frac{\partial}{\partial X}\left(K\frac{\partial \psi}{\partial X}\right) + \frac{\partial}{\partial Z}\left(K\frac{\partial \psi}{\partial Z}\right) - \frac{\partial K}{\partial Z}, \tag{1}$$

where θ is the moisture content (-), K (LT^{-1}) is the hydraulic conductivity, and ψ (L) is the suction potential.

By employing the Kirchhoff transformation

$$\Theta = \int_{-\infty}^{\psi} K ds, \tag{2}$$

where Θ is the matric flux potential (MFP), and an exponential relationship between K and ψ

$$K = K_s e^{\alpha\psi}, \quad \alpha > 0, \tag{3}$$

where α is an empirical parameter (L^{-1}) and K_s is the saturated hydraulic conductivity, the governing equation (1) can be transformed to equation

$$\frac{\partial \theta}{\partial T} = \frac{\partial^2 \Theta}{\partial X^2} + \frac{\partial^2 \Theta}{\partial Z^2} - \alpha\frac{\partial \Theta}{\partial Z}. \tag{4}$$

As discussed by Pullan [14], equation (4) may be written as

$$\frac{1}{D(\theta)}\frac{\partial \Theta}{\partial T} = \frac{\partial^2 \Theta}{\partial X^2} + \frac{\partial^2 \Theta}{\partial Z^2} - \alpha\frac{\partial \Theta}{\partial Z}, \tag{5}$$

where D ($\mathrm{L}^2\mathrm{T}^{-1}$) is the diffusivity. For mathematical convenience, it is assumed that $D(\theta)$ is a constant, and shall be written as constant D.

The horizontal and vertical components of the flux in terms of the MFP are

$$U = -\frac{\partial \Theta}{\partial X}, \tag{6}$$

and

$$V = \alpha\Theta - \frac{\partial \Theta}{\partial Z}, \tag{7}$$

respectively. The flux normal to the surface with outward pointing normal $\mathbf{n} = (n_1, n_2)$ is given by

$$F = -\frac{\partial \Theta}{\partial X}n_1 + \left(\alpha\Theta - \frac{\partial \Theta}{\partial Z}\right)n_2. \tag{8}$$

By using the definition of the MFP and the flux normal, boundary conditions described in the preceding section can be written as follows

$$\begin{aligned} F &= -v_0, \text{ on the surface of the channels,} && (9)\\ F &= 0, \text{ on the soil surface outside the channels,} && (10)\\ \frac{\partial \Theta}{\partial X} &= 0, \quad X = 0 \text{ or } X = L + D, && (11) \end{aligned}$$

and

$$\frac{\partial \Theta}{\partial X} = \frac{\partial \Theta}{\partial Z} = 0, \quad Z = \infty \text{ and } 0 \leq X \leq L + D. \tag{12}$$

Dimensionless variables are defined as

$$\begin{aligned} x &= \frac{\alpha}{2}X, \quad z = \frac{\alpha}{2}Z, \quad \Phi = \frac{\pi\Theta}{v_0 L}, \quad t = \frac{\alpha^2 D}{4}t,\\ u &= \frac{2\pi}{v_0\alpha L}U, \quad v = \frac{2\pi}{v_0\alpha L}V, \text{ and } f = \frac{2\pi}{v_0\alpha L}F. \end{aligned} \tag{13}$$

By using the dimensionless variables in (13), equation (5) becomes

$$\frac{\partial \Phi}{\partial t} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} - 2\frac{\partial \Phi}{\partial z}. \tag{14}$$

Equations (6) to (8) and dimensionless variables (13) yield

$$\begin{aligned} u &= -\frac{\partial \Phi}{\partial x}, && (15)\\ v &= 2\Phi - \frac{\partial \Phi}{\partial z}, && (16) \end{aligned}$$

and

$$f = -\frac{\partial \Phi}{\partial x}\, n_1 + \left(2\Phi - \frac{\partial \Phi}{\partial z}\right) n_2. \tag{17}$$

Now, from dimensionless variables (13) and equation (17), boundary conditions (9) to (12) can be written as

$$\begin{aligned} f &= -\frac{2\pi}{\alpha L}, \text{ on the surface of the channels,} && (18)\\ f &= 0, \text{ on the soil surface outside the channels,} && (19)\\ \frac{\partial \Phi}{\partial x} &= 0, \quad x = 0 \text{ or } x = \frac{\alpha}{2}(L + D), && (20) \end{aligned}$$

and

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial z} = 0, \quad z = \infty \text{ and } 0 \leq x \leq \frac{\alpha}{2}(L + D). \tag{21}$$

Taking Laplace transforms of equation (14) and boundary conditions (18) to (21) subject to the initial condition

$$\Phi(x, z, 0) = 0, \tag{22}$$

we obtain a new governing equation

$$s\Phi^* = \frac{\partial^2 \Phi^*}{\partial x^2} + \frac{\partial^2 \Phi^*}{\partial z^2} - 2\frac{\partial \Phi^*}{\partial z}, \tag{23}$$

with boundary conditions

$$f^* = -\frac{2\pi}{\alpha L}.\frac{1}{s}, \text{ on the surface of the channels,} \tag{24}$$

$$f^* = 0, \text{ on the soil surface outside the channels,} \tag{25}$$

$$\frac{\partial \Phi^*}{\partial x} = 0, \quad x = 0 \text{ or } x = \frac{\alpha}{2}(L + D), \tag{26}$$

and

$$\frac{\partial \Phi^*}{\partial x} = \frac{\partial \Phi^*}{\partial z} = 0, \quad z = \infty \text{ and } 0 \le x \le \frac{\alpha}{2}(L + D), \tag{27}$$

where

$$\Phi^*(x, z, s) = \int_0^\infty e^{-st}\Phi(x, z, t)\, dt, \tag{28}$$

and

$$f^* = -\frac{\partial \Phi^*}{\partial x} n_1 + \left(2\Phi^* - \frac{\partial \Phi^*}{\partial z}\right) n_2. \tag{29}$$

Making use of the transformation

$$\Phi^* = e^z \phi, \tag{30}$$

equation (23) becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = (1 + s)\phi, \tag{31}$$

and

$$f^* = \left(\phi\, n_2 - \frac{\partial \phi}{\partial n}\right) e^z. \tag{32}$$

Boundary conditions in terms of ϕ are

$$\frac{\partial \phi}{\partial n} = \frac{2\pi}{\alpha L s} e^{-z} + \phi\, n_2, \text{ on the surface of the channels,} \tag{33}$$

$$\frac{\partial \phi}{\partial n} = -\phi, \text{ on the soil surface outside the channels,} \tag{34}$$

$$\frac{\partial \phi}{\partial n} = 0, \quad x = 0 \text{ or } x = \frac{\alpha}{2}(L + D), \tag{35}$$

and

$$\frac{\partial \phi}{\partial n} = -\phi, \quad z = \infty \text{ and } 0 \leq x \leq \frac{\alpha}{2}(L + D). \tag{36}$$

The governing equation (31) is a Helmholtz equation which may be solved numerically. In this paper, a DRBEM is employed to obtain numerical solutions to equation (31) subject to boundary conditions (33) - (36). This method was initially introduced by Brebbia and Nardini [6]. This method has been used widely by numerous researchers such as Zhu et al. [19], Ang [2], and Ang and Ang [3]. An integral equation for solving equation (31) is

$$\begin{aligned} \lambda(\xi, \eta)\phi(\xi, \eta) &= \int\int_R \varphi(x, z; \xi, \eta)[(1 + s)\phi(x, z)]dxdz \\ &+ \int_C \left[\phi(x, z)\frac{\partial}{\partial n}(\varphi(x, z; \xi, \eta))\right. \\ &- \left.\varphi(x, z; \xi\eta)\frac{\partial}{\partial n}(\phi(x, z))\right] ds(x, z), \end{aligned} \tag{37}$$

where

$$\lambda(\xi, \eta) = \begin{cases} \frac{1}{2}, & (\xi, \eta) \text{ lies on a smooth part of } C \\ 1, & (\xi, \eta) \in R \end{cases}, \tag{38}$$

and

$$\varphi(x, z; \xi, \eta) = \frac{1}{4\pi} \ln[(x - \xi)^2 + (z - \eta)^2] \tag{39}$$

is the fundamental solution of the Laplace's equation.

A numerical solution to equation (31) can be obtained by using the integral equation (37). Boundary C is discretized by constant elements, and a number of interior points is chosen. Let N and M be the numbers of the elements and the chosen interior points respectively. Points $(a^{(1)}, b^{(1)})$, $(a^{(2)}, b^{(2)})$, ..., $(a^{(N)}, b^{(N)})$ be the midpoints of the line segments, and $(a^{(N+1)}, b^{(N+1)})$, $(a^{(N+2)}, b^{(N+2)})$, ..., $(a^{(N+M)}, b^{(N+M)})$ be the chosen interior points. The integral equation is reduced to the following system of linear algebraic equations

$$\begin{aligned} \lambda(a^{(n)}, b^{(n)})\phi^{(n)} &= \sum_{k=1}^{N+M} \nu^{(nk)}[(1 + s)\phi^{(k)}] \\ &+ \sum_{m=1}^{N} [\phi^{(k)}\mathfrak{F}_1^{(m)}(a^{(n)}, b^{(n)}) - p^{(k)}\mathfrak{F}_2^{(m)}(a^{(n)}, b^{(n)})], \\ & n = 1, 2, \cdots, N + M. \end{aligned} \tag{40}$$

where

$$\nu^{(nk)} = \sum_{i=1}^{N+M} \Upsilon(a^{(n)}, b^{(n)}; a^{(i)}, b^{(i)})\omega^{(ik)}, \tag{41}$$

$$\mathfrak{F}_1^{(m)}(a^{(n)}, b^{(n)}) = \int_{C^{(m)}} \varphi(x, z; a^{(n)}, b^{(n)})ds(x, z), \tag{42}$$

and

$$\mathfrak{F}_2^{(m)}(a^{(n)}, b^{(n)}) = \int_{C^{(m)}} \frac{\partial}{\partial n}(\varphi(x, z; a^{(n)}, b^{(n)}))ds(x, z). \tag{43}$$

Function $\Upsilon(a^{(n)}, b^{(n)}; a^{(i)}, b^{(i)})$ and $\omega^{(ik)}$ are

$$\begin{aligned} \Upsilon(a^{(n)}, b^{(n)}; a^{(i)}, b^{(i)}) &= \lambda(a^{(n)}, b^{(n)})\chi(a^{(n)}, b^{(n)}; a^{(i)}, b^{(i)}) \\ &+ \sum_{j=1}^{N} \frac{\partial}{\partial n}(\chi(x, z; a^{(i)}, b^{(i)})) \Big|_{(x,z)=(a^{(j)},b^{(j)})} \mathfrak{F}_1^{(j)}(a^{(n)}, b^{(n)}) \\ &+ \sum_{j=1}^{N} \chi(a^{(j)}, b^{(j)}; a^{(i)}, b^{(i)})\mathfrak{F}_2^{(j)}(a^{(n)}, b^{(n)}), \end{aligned} \tag{44}$$

and

$$[\omega^{(ik)}] = [\rho(a^{(k)}, b^{(k)}; a^{(i)}, b^{(i)})]^{-1}, \tag{45}$$

where

$$\begin{aligned} \rho(x, z; a^{(i)}, b^{(i)}) = 1 + ((x - a^{(i)})^2 + (z - b^{(i)})^2) \\ +((x - a^{(i)})^2 + (z - b^{(i)})^2)^{3/2}, \end{aligned} \tag{46}$$

and

$$\begin{aligned} \chi(x, z; a^{(i)}, b^{(i)}) &= \frac{1}{4}[(x - a^{(i)})^2 + (z - b^{(i)})^2] + \frac{1}{16}[(x - a^{(i)})^2 + (z - b^{(i)})^2]^2 \\ &+ \frac{1}{25}[(x - a^{(i)})^2 + (z - b^{(i)})^2]^{5/2}. \end{aligned} \tag{47}$$

Equation (40) may be solved to obtain the values of ϕ at the collocation point. By using these values of ϕ, the numerical value of ϕ at any point in the domain can be obtained using equation (40).

To compute numerical values of the dimensionless MFP, we first employ equation (30) to obtain numerical values of Φ^*, and then use the Stehfest formula to determine the numerical values of their inverse Laplace transform. The formula is as follows (see [10])

$$\Phi(x, z, t) \simeq \frac{\log 2}{t} \sum_{n=1}^{2N} K_n \Phi^*(x, z, s_n), \tag{48}$$

where

$$s_n = n\frac{\log 2}{t}, \tag{49}$$

$$K_n = (-1)^{(n+N)} \sum_{m=(n+1)/2}^{\min(n,N)} \frac{m^N (2m)!}{(N - m)!\, m!\, (m - 1)!\, (n - m)!\, (2m - n)!} \tag{50}$$

and N is a positive integer.

4 Results and discussion

The method described in Section 3 is tested through a problem involving time-dependent infiltration from periodic trapezoidal channels into soil of the homogeneous pima clay loam (PCL) type. A trapezoidal channel is chosen because it is more commonly used by farmers, especially in developing countries. The value of the experimental parameter α for PCL is 0.014 cm^{-1}. This value is as reported by Amozegar-Fard et al [1] and Bresler [7]. We set $L = D = 50$ cm, and the width and the depth of the channel is $4L/\pi$ and $3L/\pi$ respectively.

The DRBEM is employed to obtain numerical solutions to equation (31) subject to boundary conditions (33) to (36). To employ the DRBEM, the domain must be bounded by a simple closed curve. An appropriate depth for boundary conditions to be applied without significant impact to values of ϕ in the domain is $z = 4$. Therefore, the domain is set to be between $z = 0$ and $z = 4$. The boundary is divided into 404 constant elements, and 892 interior points are chosen.

After obtaining ϕ, the values of Φ^* can be computed using equation (30). Finally, the dimensionless MFP, Φ, are obtained numerically by employing the Stehfest formula (48) with $N = 3$. Results obtained are presented in Figures 2 and 3.

Figure 2 contains a series of two-dimensional cross-section plots that show the distribution of Φ, over a region bounded by $0 \leq x \leq 0.7$ and $0 \leq z \leq 1.4$ for several different levels of time, that is $t = 0.6$, $t = 0.8$, $t = 1$, $t = 2$, $t = 3$, and $t = \infty$. The plots illustrate the distribution of Φ over the region as the dimensionless time t increased. From $t = 0.6$ to $t = 0.8$, it can be seen clearly that the distribution of Φ changes over the region. This also occur from $t = 0.8$ to $t = 1$, and from $t = 1$ to $t = 2$. These mean that there may significant increase in water content in the soil from $t = 0.6$ to $t = 0.8$, as well as from $t = 0.8$ to $t = 1$ and from $t = 1$ to $t = 2$.

From $t = 2$ to $t = 3$, it seems that there are no significant increase in the distribution of Φ at the surface of the channel, but at other locations, significant changes are observed. After $t = 3$, the distribution of Φ remains more or less constant over the region. These observations indicate that points at some level achieve maximum water content earlier than those deeper. These results are physically meaningful, as irrigation water passes through a level of soil first, before going deeper. At the shallower level, some of the water is absorbed, and then the rest moves through to deeper levels.

Figure 3 shows the variation of Φ as t increases at four chosen points. These points are selected such that two points are located at $z = 0.2$ and two other are at $z = 1.0$. In the figure, graphs (a) and (d) are graphs of Φ at $(0.1, 0.2)$ and $(0.6, 0.2)$ respectively. Graphs of Φ at $(0.1, 1.0)$ and $(0.6, 1.0)$ are labelled as (b) and (c). It can be seen that the two graphs, (b) and (c), behave in a similar fashion. On the other hand, (a) and (d) share a similar trend different from the other two, except for $0.6 < t < 0.8$. This means that at the same level of z, the rate of change of Φ for $0.8 < t < 5$ is almost equal for all points at a constant depth. This implies that at any time t, $0.8 \leq t \leq 5$, the amount of water absorbed at one point is about

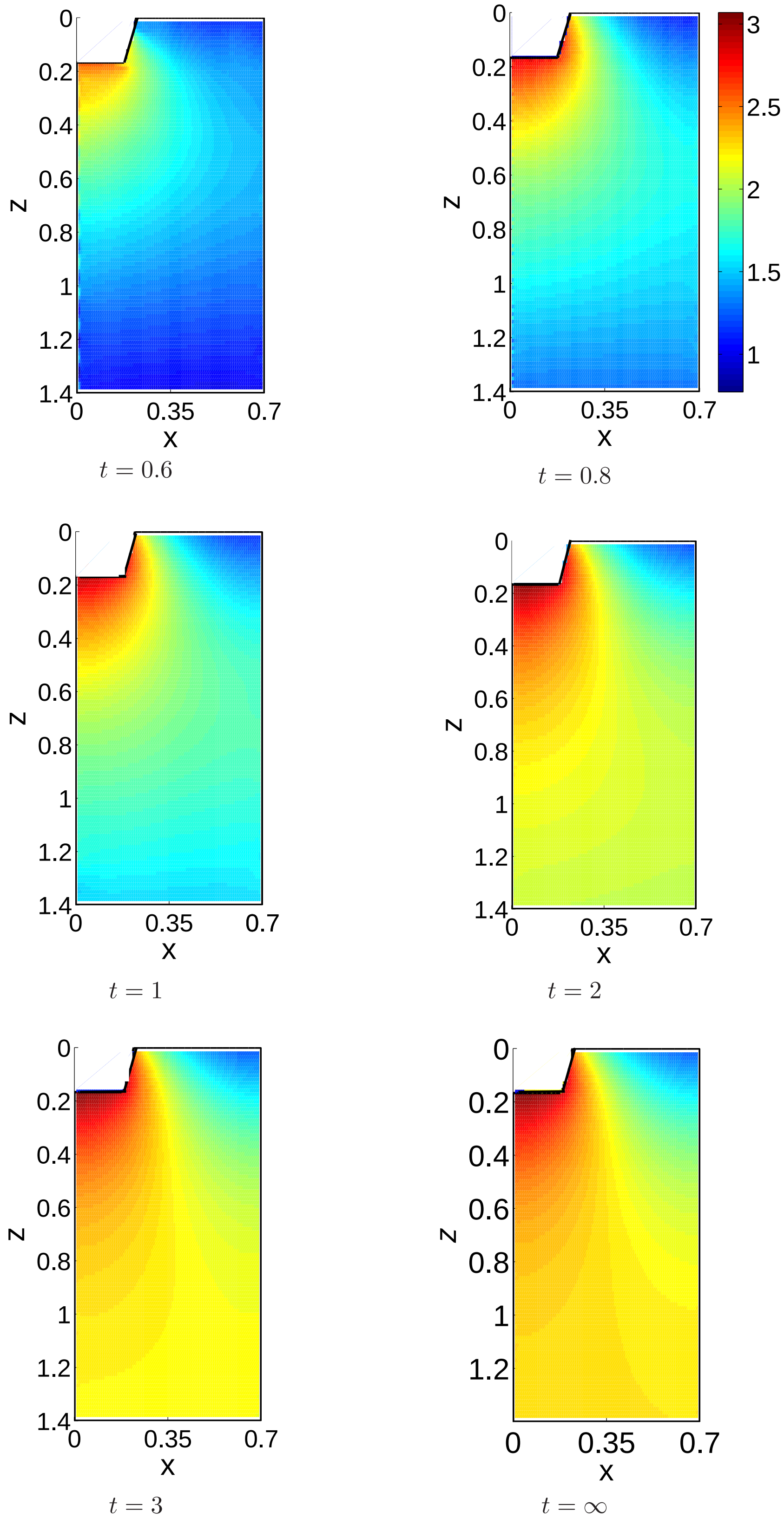

Figure 2: Two-dimensional cross-section plots of Φ at $t = 0.6$, $t = 0.8$, $t = 1$, $t = 2$, $t = 3$, and $t = \infty$

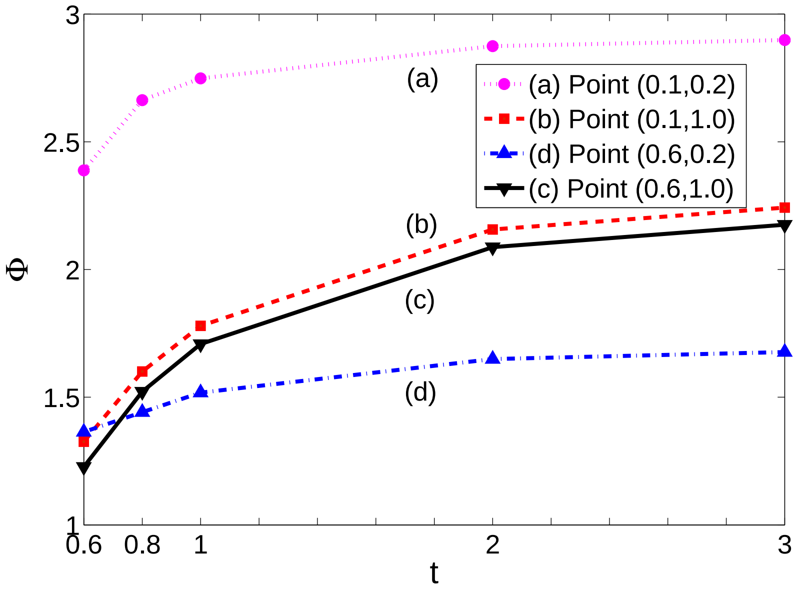

Figure 3: Variation of Φ at some points as t increases

the same as that at other points at the same level of z.

For t=0.6, there may be points where the rate of changes of Φ are different from those at some other points at the same level of z. This result is expected. The value of Φ at $t = 0$, when there is no water infiltrating the soil, is zero everywhere. At some later value of t, for instance at $t = 1$, and some level of z, Φ is likely to vary and be different at different points along a fixed horizontal level. This implies that the rate of change of Φ may change over time.

Although not presented in the graphs shown in Figure 3, it is observed at in all cases, Φ increases at the start of the infiltration process but gradually tapers off to a constant maximum value. This occurs at all values of z although there are some differences in the times at which Φ levels off. From Figure 3, it can be seen that at z= 0.2, Φ increases fairly quickly before $t = 1$ when it begins to level off. In contrast, at $z = 1.0$, Φ rises rapidly even after $t = 1$, levelling off only at a later time.

The results show that water content in soil increases over time until they reach their maximum levels. The results also indicate that at any given point in time, the amount of water absorbed by the soil at a fixed depth may be the same in all horizontal direction, provided sufficient time is given for infiltration. The results also show that a point at a shallow level of soil depth reaches its maximum water content more rapidly than those at deeper levels.

5 Concluding remarks

A problem involving time-dependent infiltration from periodic trapezoidal channels has been solved by applying a set of transformations, including the Laplace transform, and a DRBEM. The Laplace transform is inverted using the Stehfest formula to obtain numerical solutions of the dimensionless MFP.

The results obtained indicate the difference in time needed to approach the steady state value of the dimensionless MFP. Points at a level need more time than that at shallower levels, which means that points at shallower levels of soil depth reach their maximum water contents faster than those deeper. Studies on the water contents at top level of soil should be explored in the future research by including the effect of water uptake by plant roots, as the root zone is normally at the top soil.

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