Boundary integral equations for plane orthotropic bodies and exterior regions

G. Szeidl and J. Dudra
University of Miskolc, Department of Mechanics
3515 Miskolc-Egyetemváros, Hungary

Abstract

Assuming linear displacements and constant strains and stresses at infinity, we reformulate the equations of the direct boundary element method for plane problems of elasticity. We consider a body made of orthotropic material. The reformulated equations make it possible to attack plane problems on exterior regions without replacing the region by a bounded one.

1 Introduction

As is well known a large literature studies plane problems for orthotropic bodies, including for instance [1, 2, 3], as well as the books [4, 5] and the references therein. However, as explained by Schiavone [6], the standard formulation for exterior regions has the disadvantage that it is impossible to prescribe a constant stress state at infinity.

The reason is that an assumption about the far field pattern of the displacements is needed in order to establish an appropriate Betti's formula and to prove uniqueness and existence for the exterior Dirchlet and Neuman problems. Unfortunately, this assumption excludes those problems from the theory for which the displacements are linear while the strains and stresses are constant at infinity.

To make progress on plane problems with such displacements, we note that if the direct formulation reproduces this displacement field, then the resulting strain and stress conditions must also be constant at infinity. Consequently, plane problems for the exterior regions can be attacked without replacing the region by a bounded one. The work in [7] and [8] presents such direct formulations by assuming constant strains and stresses at infinity for an isotropic body. For exterior regions [7] reformulates the classical approach to plane problems. For the same class of problems but in a dual formulation [8] sets up the equations of the direct method in terms of stress functions of order one.

The present paper is an attempt to clarify how the formulation changes if we apply the ideas presented in paper [7] to orthotropic bodies.

2 Basic Equations in Generalized Plane Strain

Throughout this paper x_1 and x_2 are rectangular Cartesian coordinates, referred to an origin O. Greek subscripts are assumed to have the range (1,2), summation over repeated subscripts is implied. The doubly connected exterior region under

consideration is denoted by A_e and is bounded by the contour $\mathcal{L}_o$. We stipulate that the contour admits a nonsingular parametrization in terms of its arc length s. The outer normal is denoted by n_π. In accordance with the notations introduced, $\delta_{\kappa\lambda}$ is the Kronecker symbol, ∂_α stands for the derivatives taken with respect to x_α and $\epsilon_{3\kappa\lambda}$ is the permutation symbol. Assuming plane problems let u_κ, $e_{\kappa\lambda}$ and $t_{\kappa\lambda}$ be the displacement field and the in plane components of strain and stress, respectively. For orthotropic bodies the material constants are denoted by s_{11}, $s_{12} = s_{21}$, s_{22} and s_{66}.

For homogenous and orthotropic material the plane problem of classical elasticity is governed by the kinematic equations

$$e_{\rho\lambda} = \frac{1}{2}(\partial_\rho u_\lambda + \partial_\lambda u_\rho), \tag{1}$$

Hook's law

$$\begin{aligned} t_{11} &= c_{11}e_{11} + c_{12}e_{22} \ , \\ t_{22} &= c_{12}e_{11} + c_{22}e_{22} \ , \\ t_{12} &= t_{21} = 2c_{66}e_{12} \ , \end{aligned} \tag{2}$$

where

$$c_{11} = \frac{s_{22}}{d}, \quad c_{12} = c_{21} = -\frac{s_{12}}{d}, \quad c_{22} = \frac{s_{11}}{d}, \quad c_{66} = \frac{1}{s_{66}}, \quad d = s_{11}s_{22} - s_{12}^2 \ ; \tag{3}$$

and the equilibrium equations

$$t_{\rho\lambda}\partial_\lambda + b_\rho = 0 \tag{4}$$

which should be complemented with appropriate boundary conditions not detailed here since they play no role in the present investigations. The basic equation for u_λ takes the form

$$\mathcal{D}_{\rho\lambda}u_\lambda + b_\rho = 0 \ , \tag{5a}$$

where the differential operator $\mathcal{D}_{\rho\lambda}$ has the form

$$[\mathcal{D}_{\rho\lambda}] = \begin{bmatrix} c_{11}\partial_1^2 + c_{66}\partial_2^2 & (c_{12} + c_{66})\,\partial_1\partial_2 \\ (c_{21} + c_{66})\,\partial_2\partial_1 & c_{22}\partial_2^2 + c_{66}\partial_1^2 \end{bmatrix} \ . \tag{5b}$$

Let $Q(\xi_1, \xi_2)$ and $M\,(x_1, x_2)$ be two points in the plane (the source point and the field point). We shall assume temporarily that the point Q is fixed. The distance between Q and M is R, the position vector of M relative to Q is r_κ. The small circle as a subscript (for instance $M_\circ$ or $Q_\circ$) indicates that the corresponding points, i.e., Q or M are taken on the contour.

It is obvious that

$$r_\alpha(M, Q) = x_\alpha(M) - \xi_\alpha(Q) = x_\alpha - \xi_\alpha \ . \tag{6}$$

Let us introduce the following notations

$$\lambda_1 + \lambda_2 = (2s_{12} + s_{66})/s_{22}\,, \tag{7}$$

$$\lambda_1\lambda_2 = s_{11}/s_{22}\,, \tag{8}$$

$$A_\alpha = s_{12} - \lambda_\alpha s_{22}\,, \tag{9}$$

$$\rho_\alpha^2 = \lambda_\alpha r_1^2 + r_2^2\,, \tag{10}$$

$$D = \frac{1}{2\pi\,(\lambda_1 - \lambda_2)\,s_{22}}\,. \tag{11}$$

For our later considerations we note that equations (7) and (8) imply

$$\lambda_{1,2} = \frac{2s_{21} + s_{66}}{2s_{22}} \pm \sqrt{(\frac{2s_{21} + s_{66}}{2s_{22}})^2 - \frac{s_{11}}{s_{22}}}\,. \tag{12}$$

The well known singular fundamental solutions for the basic equation (5a) [1, 3] are given by the formulas

$$\begin{aligned}
U_{11}(M,Q) &= D\left(\sqrt{\lambda_1}A_2^2\ln\rho_1 - \sqrt{\lambda_2}A_1^2\ln\rho_2\right)\,,\\
U_{12}(M,Q) &= DA_1A_2\arctan\frac{\left(\sqrt{\lambda_1}-\sqrt{\lambda_2}\right)r_1r_2}{\sqrt{\lambda_1}\sqrt{\lambda_2}r_1^2 + r_2^2}\,,\\
U_{21}(M,Q) &= U_{12}(M_\circ,Q)\,,\\
U_{22}(M,Q) &= -D\left(\frac{A_1^2\ln\rho_1}{\sqrt{\lambda_1}} - \frac{A_2^2\ln\rho_2}{\sqrt{\lambda_2}}\right)
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
T_{11}(M,Q) &= D\left[\frac{\sqrt{\lambda_2}A_1}{\rho_2^2} - \frac{\sqrt{\lambda_1}A_2}{\rho_1^2}\right](r_1n_1 + r_2n_2)\,,\\
T_{12}(M,Q) &= D\left\{\left(\frac{\sqrt{\lambda_1}A_1}{\rho_1^2} - \frac{\sqrt{\lambda_2}A_2}{\rho_2^2}\right)r_1n_2 - \left(\frac{1}{\sqrt{\lambda_1}}\frac{A_1}{\rho_1^2} - \frac{1}{\sqrt{\lambda_2}}\frac{A_2}{\rho_2^2}\right)r_2n_1\right\}\,,\\
T_{21}(M,Q) &= D\left\{\left(\frac{\lambda_1\sqrt{\lambda_1}A_2}{\rho_1^2} - \frac{\lambda_2\sqrt{\lambda_2}A_1}{\rho_2^2}\right)r_1n_2 - \left(\frac{\sqrt{\lambda_1}A_2}{\rho_1^2} - \frac{\sqrt{\lambda_2}A_1}{\rho_2^2}\right)r_2n_1\right\}\,,\\
T_{22}(M,Q) &= D\left[\frac{\sqrt{\lambda_1}A_1}{\rho_1^2} - \frac{\sqrt{\lambda_2}A_2}{\rho_2^2}\right](r_1n_1 + r_2n_2)\,,
\end{aligned} \tag{14}$$

where

$$\mathfrak{u}_\lambda(M) = U_{\lambda\kappa}(M,Q)e_\kappa(Q) \qquad \text{and} \qquad t_\lambda(M) = T_{\lambda\kappa}(M,Q)e_\kappa(Q)$$

are the displacement vector and stress vector on a line element with a normal $n_\lambda = n_\lambda(M)$ to it at the point M due to the force $e_\kappa = e_\kappa(Q)$ at Q.

3 Basic formulas for exterior regions

Figure 1 depicts a triple connected region A'_e bounded by the contours $\mathcal{L}_o$, $\mathcal{L}_\varepsilon$ and the circle $\mathcal{L}_R$ with radius ${}_eR$ and center at O. Here $\mathcal{L}_\varepsilon$ is the contour of the neighborhood A_ε of Q with radius R_ε while ${}_eR$ is sufficiently large so that the region bounded by $\mathcal{L}_R$ covers both $\mathcal{L}_0$, and $\mathcal{L}_\varepsilon$. If ${}_eR \to \infty$ and $R_\varepsilon \to 0$ then clearly $A'_e \to A_e$.

Let $u_\kappa(M)$ and $g_\kappa(M)$ be sufficiently smooth – continuously differentiable at least twice – but otherwise arbitrary displacement fields on A_e. The stresses obtained from these displacement fields are denoted by $t_{\lambda\kappa}\left[u_\rho(M)\right]$ and $t_{\lambda\kappa}\left[g_\rho(M)\right]$ respectively.

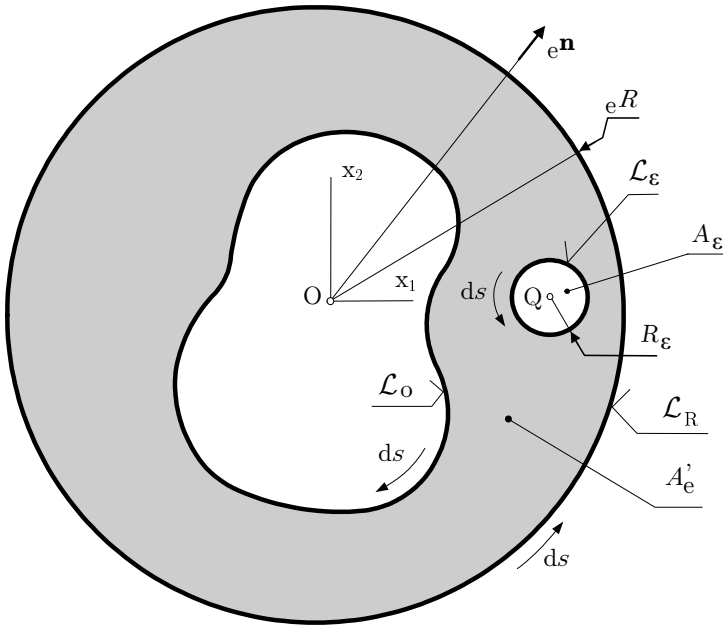

Figure 1:

Equation

$$\int_{A'_e}\left[u_\lambda(M)\left(\mu\overset{M}{\mathcal{D}}_{\lambda\sigma}g_\sigma(M)\right)-g_\lambda(M)\left(\mu\overset{M}{\mathcal{D}}_{\lambda\sigma}u_\sigma(M)\right)\right]\mathrm{d}A_M\;=$$
$$=\oint_{\mathcal{L}_o}\left[u_\lambda(M_\circ)t_{\lambda\kappa}\left[g_\rho(M_\circ)\right]n_\kappa(M_\circ)-g_\lambda(M_\circ)t_{\lambda\kappa}\left[u_\rho(M_\circ)\right]n_\kappa(M_\circ)\right]\mathrm{d}s_{M_\circ}$$
$$+\oint_{\mathcal{L}_\varepsilon}\left[u_\lambda(M_\circ)t_{\lambda\kappa}\left[g_\rho(M_\circ)\right]n_\kappa(M_\circ)-g_\lambda(M_\circ)t_{\lambda\kappa}\left[u_\rho(M_\circ)\right]n_\kappa(M_\circ)\right]\mathrm{d}s_{M_\circ}$$
$$+\oint_{\mathcal{L}_R}\left[u_\lambda(M_\circ)t_{\lambda\kappa}\left[g_\rho(M_\circ)\right]n_\kappa(M_\circ)-g_\lambda(M_\circ)t_{\lambda\kappa}\left[u_\rho(M_\circ)\right]n_\kappa(M_\circ)\right]\mathrm{d}s_{M_\circ}\;,\quad(15)$$

in which M over a letter denotes derivatives taken with respect to the point coordinates M and $n_\kappa(M_\circ)$ is the outward normal, is the primal Somigliana identity applied to the triple connected region A'_e.

Let $g_\lambda(Q) = U_{\lambda\kappa}(M,Q)e_\kappa(Q)$, which is a non singular elastic state of the plane in A'_e. We regard $u_\lambda(M)$ as a different elastic state in the region A_e. Further we assume that $u_\lambda(M)$ has the far field pattern (asymptotic behavior)

$$\tilde{u}_\kappa(M) = c_\kappa + \varepsilon_{3\rho\kappa}x_\rho\omega + e_{\kappa\beta}(\infty)x_\beta \tag{16}$$

as x_β or equivalently M tends to infinity. Here c_κ is a translation, ω is a rotation in finite, $c_\kappa + \varepsilon_{3\rho\kappa}x_\rho\omega$ is the corresponding rigid body motion, $e_{\kappa\beta}(\infty)$ is a constant strain tensor at infinity and $e_{\kappa\beta}(\infty)x_\beta$ is the corresponding displacement field.

The stresses induced by the strains $e_{\kappa\beta}(\infty)$ can be obtained by the Hooke law:

$$\begin{aligned} t_{11}(\infty) &= c_{11}e_{11}(\infty) + c_{12}e_{22}(\infty)\ , \\ t_{22}(\infty) &= c_{12}e_{11}(\infty) + c_{22}e_{22}(\infty)\ , \\ t_{12}(\infty) &= t_{21}(\infty) = 2c_{66}e_{12}(\infty)\ , \end{aligned} \tag{17}$$

Substituting the above quantities into the Somigliana identity, we obtain

$$\begin{aligned} \int_{A'_e} \left[u_\lambda(M) \left(\mu \overset{M}{\mathcal{D}}_{\lambda\sigma} U_{\sigma\kappa}(M,Q) \right) - \left(\mu \overset{M}{\mathcal{D}}_{\lambda\sigma} u_\sigma(M) \right) U_{\lambda\kappa}(M,Q) \right] \mathrm{d}A_M\, e_\kappa(Q) = \\ = \oint_{\mathcal{L}_o} \left[u_\lambda(M_\circ) T_{\lambda\kappa}(M_\circ,Q) - t_\lambda(M_\circ) U_{\lambda\kappa}(M_\circ,Q) \right] \mathrm{d}s_{M_\circ}\, e_\kappa(Q) + \\ + \oint_{\mathcal{L}_\varepsilon} \left[u_\lambda(M_\circ) T_{\lambda\kappa}(M_\circ,Q) - t_\lambda(M_\circ) U_{\lambda\kappa}(M_\circ,Q) \right] \mathrm{d}s_{M_\circ}\, e_\kappa(Q) \\ + \oint_{\mathcal{L}_R} \left[u_\lambda(M_\circ) T_{\lambda\kappa}(M_\circ,Q) - t_\lambda(M_\circ) U_{\lambda\kappa}(M_\circ,Q) \right] \mathrm{d}s_{M_\circ}\, e_\kappa(Q) \end{aligned} \tag{18}$$

since $t_{\lambda\kappa}\left[u_\rho(M_\circ)\right] n_\kappa(M_\circ) = t_\lambda(M_\circ)$ is the stress on the contour and obviously

$$t_{\lambda\kappa}\left[g_\rho(M_\circ)\right] n_\kappa(M_\circ) = T_{\lambda\kappa}(M_\circ,Q)e_\kappa(Q).$$

In the sequel we shall assume that there are no body forces. This assumption has no effect on the result we will obtain.

It is clear that one can omit $e_\kappa(Q)$. Regarding the equation obtained by omitting $e_\kappa(Q)$, our goal is to compute its limit as $R_\varepsilon \longrightarrow 0$ and ${}_eR \longrightarrow \infty$. As is well known, the left hand side vanishes under the conditions detailed above(see [4]), and hence

$$\oint_{\mathcal{L}_o} \cdots + \lim_{R_\varepsilon \longrightarrow 0} \oint_{\mathcal{L}_\varepsilon} \cdots = u_\kappa(Q) + \oint_{\mathcal{L}_o} \left[u_\lambda(\overset{o}{M}) T_{\lambda\kappa}(M_\circ,Q) - t_\lambda(M_\circ) U_{\lambda\kappa}(M_\circ,Q) \right] \mathrm{d}s_{M_\circ}\ .$$

Consequently

$$\begin{aligned} u_\kappa(Q) = \lim_{{}_eR \longrightarrow \infty} \oint_{\mathcal{L}_R} \left[t_\lambda(M_\circ) U_{\lambda\kappa}(M_\circ,Q) - u_\lambda(M_\circ) T_{\lambda\kappa}(M_\circ,Q) \right] \mathrm{d}s_{M_\circ} + \\ + \oint_{\mathcal{L}_o} \left[t_\lambda(M_\circ) U_{\lambda\kappa}(M_\circ,Q) - u_\lambda(M_\circ) T_{\lambda\kappa}(M_\circ,Q) \right] \mathrm{d}s_{M_\circ}\ . \end{aligned} \tag{19}$$

In order to establish the first Somigliana formula for the exterior region, we need to find the limit of the first integral on the right hand side.

4 Somigliana formulas modified for exterior regions

Our goal in this section is to prove that

$$I_\kappa = \lim_{{}_eR \longrightarrow \infty} \oint_{\mathcal{L}_R} [t_\lambda(M_\circ) U_{\lambda\kappa}(M_\circ, Q) - u_\lambda(M_\circ) T_{\lambda\kappa}(M_\circ, Q)] \, \mathrm{d}s_{M_\circ} = \\ = c_\kappa + \varepsilon_{3\rho\kappa} \xi_\rho \omega + e_{\kappa\beta}(\infty) \xi_\beta = \tilde{u}_\kappa(Q) \, . \quad (20)$$

The proof is based on the first Somigliana formula for inner regions and requires simple tools only[1].

Consider the simple connected region A_R bounded by the circle $\mathcal{L}_R$ with radius ${}_eR$ and center at O. Assume that Q is in the interior of the region. The displacement field $\tilde{u}(M)$ is an elastic state of A_R with no body forces. The corresponding strain and stress tensors are denoted by $\tilde{e}_{\kappa\lambda}(M)$ and $\tilde{t}_{\kappa\lambda}(M)$, respectively. The stress vector on the boundary is $\tilde{t}_\kappa = \tilde{t}_{\kappa\lambda}(M_o) \, n_\kappa(M_o)$. Since $\tilde{u}(M)$ is an elastic state of the orthotropic inner region A_R, the first Somigliana formula implies

$$\tilde{u}_\kappa(Q) = \oint_{\mathcal{L}_R} \left[\tilde{t}_\lambda(M_\circ) U_{\lambda\kappa}(M_\circ, Q) - \tilde{u}_\lambda(M_\circ) T_{\lambda\kappa}(M_\circ, Q) \right] \mathrm{d}s_{M_\circ} \, . \quad (21)$$

Now consider the displacement field

$$\tilde{u}_\lambda(M) = c_\kappa + \varepsilon_{3\rho\kappa} x_\rho \omega + e_{\kappa\beta}(\infty) x_\beta \quad (22)$$

which describes the asymptotic behavior of $u_\lambda(M)$ in the exterior region A_e. It can be checked that this displacement field is an elastic state of the entire plane, including the region A_R, for which the stress tensor is constant, i.e., $\tilde{t}_{\kappa\lambda}(M) = \tilde{t}_{\kappa\lambda}(\infty)$. Given an elastic state of the plane, combining equations (21) and (22) yields

$$\oint_{\mathcal{L}_R} \left[\tilde{t}_\lambda(M_\circ) U_{\lambda\kappa}(M_\circ, Q) - \tilde{u}_\lambda(M_\circ) T_{\lambda\kappa}(M_\circ, Q) \right] \mathrm{d}s_{M_\circ} = \\ = \tilde{u}_\lambda(Q) = c_\kappa + \varepsilon_{3\rho\kappa} \xi_\rho \omega + e_{\kappa\beta}(\infty) \xi_\beta \, . \quad (23)$$

Combining this with the limits

$$\lim_{{}_eR \longrightarrow \infty} u_\lambda(M) = \tilde{u}_\lambda \qquad \text{and} \qquad \lim_{{}_eR \longrightarrow \infty} t_\lambda(M) = \tilde{t}_\lambda \, , \quad (24)$$

[1] We would like to thank one of the referees for suggesting this simple derivation which replaces our more complicated argument.

and equation (23), we find that the limit of the integral in question in equation (20) is

$$\lim_{e R\longrightarrow\infty}\oint_{\mathcal{L}_R}\left[t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ,Q)-u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ,Q)\right]\mathrm{d}s_{M_\circ}= \\ =\tilde{u}_\kappa(Q)=c_\kappa+\varepsilon_{3\rho\kappa}\xi_\rho\omega+e_{\kappa\beta}(\infty)\xi_\beta\ . \quad (25)$$

Consequently, the first and modified Somigliana formulas immediately follow from equations (19) and (20):

$$u_\kappa(Q)=\tilde{u}_\kappa(Q)+\oint_{\mathcal{L}_o}\left[t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ,Q)-u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ,Q)\right]\mathrm{d}s_{M_\circ} \\ Q\in A_e \quad (26)$$

If $Q=Q_\circ$ is on $\mathcal{L}_o$, our previous logic about the integral taken on $\mathcal{L}_R$ is unaffected. Consequently

$$C_{\kappa\rho}u_\rho(Q_\circ)=\tilde{u}_\kappa(Q_\circ)+ \\ +\oint_{\mathcal{L}_o}\left[t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ,Q_\circ)-u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ,Q_\circ)\right]\mathrm{d}s_{M_\circ}\quad Q=Q_\circ\in\mathcal{L}_o \quad (27)$$

where $C_{\kappa\rho}=\delta_{\kappa\rho}/2$ if the contour is smooth at $Q_\circ$. This is the integral equation of the direct method, or the second Somigliana formula for exterior regions.

If Q is inside the contour $\mathcal{L}_o$, i.e., in region A_i, then it is easy to show that

$$0=e_{\kappa\beta}(\infty)\xi_\beta(Q)+\oint_{\mathcal{L}_o}\left[t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ,Q)-u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ,Q)\right]\mathrm{d}s_{M_\circ} \\ Q=Q_\circ\in A_i \quad (28)$$

which is the third Somigliana formula for exterior regions.

Using the formulae set up for the strains in Appendix A and the Hooke law (17) we can calculate the stresses:

$$t_{\alpha\beta}(Q)=t_{\alpha\beta}(\infty)+\oint_{\mathcal{L}_o}t_\lambda(M_\circ)\hat{\mathcal{D}}_{\lambda\alpha\beta}(M_\circ,Q)\,\mathrm{d}s_{M_\circ}- \\ -\oint_{\mathcal{L}_o}u_\lambda(M_\circ)\mathcal{S}_{\lambda\alpha\beta}(M_\circ,Q)\,\mathrm{d}s_{M_\circ}\qquad Q\in A_e \quad (29)$$

where

$$\begin{aligned}
&\hat{\mathcal{D}}_{\lambda11}=c_{11}\mathcal{D}_{\lambda11}+c_{12}\mathcal{D}_{\lambda22}\ , \quad &&\hat{\mathcal{S}}_{\lambda11}=c_{11}\mathcal{S}_{\lambda11}+c_{12}\mathcal{S}_{\lambda22}\ ,\\
&\hat{\mathcal{D}}_{\lambda11}=c_{12}\mathcal{D}_{\lambda11}+c_{22}\mathcal{D}_{\lambda22}\ , \quad &&\hat{\mathcal{S}}_{\lambda11}=c_{12}\mathcal{S}_{\lambda11}+c_{22}\mathcal{S}_{\lambda22}\ ,\\
&\hat{\mathcal{D}}_{\lambda12}=2c_{66}\mathcal{D}_{\lambda12}=\hat{\mathcal{D}}_{\lambda21}\ , \quad &&\hat{\mathcal{S}}_{\lambda12}=2c_{66}\mathcal{S}_{\lambda12}=\hat{\mathcal{S}}_{\lambda21}\ .
\end{aligned} \quad (30)$$

5 Behavior at infinity

Our goal in this section is to compute the limit of representation (26) as $Q \to \infty$. This will lead to a characterization of the asymptotic behavior of $u_\kappa(Q)$. If this behavior is the same what we have assumed, i.e., if the limit coincides with (16), then we confirm that the results of the previous section are correct. It is clear from representation (26) that it is sufficient to show that the following relations hold

$$\lim_{Q(\xi_1,\xi_2)\longrightarrow\infty} \oint_{\mathcal{L}_o} t_\lambda(M_\circ) U_{\lambda\kappa}(M_\circ, Q) \mathrm{d}s_{M_\circ} = 0 \ ,$$

$$\lim_{Q(\xi_1,\xi_2)\longrightarrow\infty} \oint_{\mathcal{L}_o} u_\lambda(M_\circ) T_{\lambda\kappa}(M_\circ, Q) \mathrm{d}s_{M_\circ} = 0 \ . \quad (31)$$

In order to find the limit of the above integrals we have to set up asymptotic relations for the fundamental solutions $U_{\lambda\kappa}(M_\circ, Q)$ and $T_{\lambda\kappa}(M_\circ, Q)$ if $Q \longrightarrow \infty$.

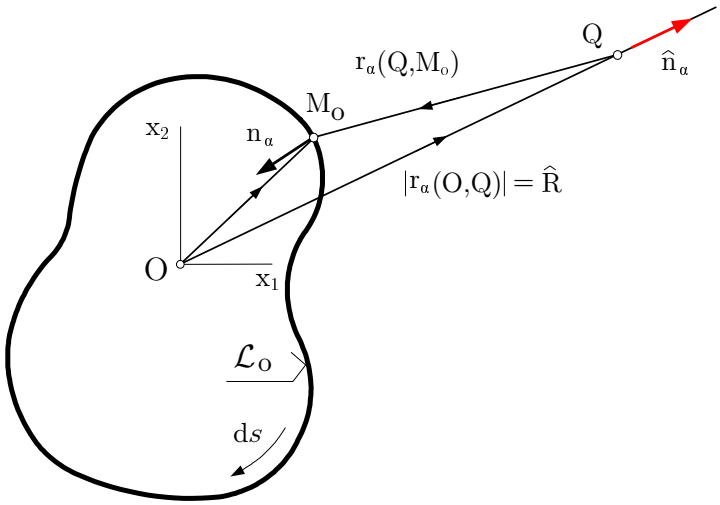

Figure 2:

Using the notations introduced in Figure 2. as well as equations (6) and (10) we have

$$r_\alpha(\overset{o}{M}, Q) = x_\alpha(\overset{o}{M}) - \xi_\alpha(Q) = x_\alpha - \xi_\alpha = -\hat{R}\left(\hat{n}_\alpha - \frac{x_\alpha}{\hat{R}}\right) \approx -\hat{R}\,\hat{n}_\alpha \quad |\hat{n}_\alpha| = 1 \ , \quad (32a)$$

$$\rho_\alpha = \sqrt{\lambda_\alpha r_1^2 + r_2^2} \approx \hat{R}\sqrt{\lambda_\alpha \hat{n}_1^2 + \hat{n}_2^2} \ , \quad (32b)$$

$$\ln \rho_\alpha \approx \ln \hat{R} + \frac{1}{2} \ln\left(\lambda_\alpha \hat{n}_1^2 + \hat{n}_2^2\right) \ . \quad (32c)$$

Substituting equations (32a,b,c) into (13) then performing some manipulations, we obtain the following asymptotic relations for the fundamental solution of order one:

$$U_{11}(M_\circ, Q) \approx D\left[\sqrt{\lambda_1}A_2^2 - \sqrt{\lambda_2}A_1^2\right]\ln\hat{R}\,, \tag{33a}$$

$$U_{12}(M_\circ, Q) = U_{21}(M_\circ, Q) \approx DA_1A_2\arctan\frac{\left(\sqrt{\lambda_1}-\sqrt{\lambda_2}\right)\hat{n}_1\hat{n}_2}{\sqrt{\lambda_1}\sqrt{\lambda_2}\hat{n}_1^2+\hat{n}_2^2}\,, \tag{33b}$$

$$U_{22}(M_\circ, Q) \approx -D\left[\frac{A_1^2}{\sqrt{\lambda_1}} - \frac{A_2^2}{\sqrt{\lambda_2}}\right]\ln\hat{R}\,. \tag{33c}$$

It is obvious that asymptotically $U_{\lambda\kappa}(M_\circ, Q) \approx U_{\lambda\kappa}(Q)$, i.e., the kernel in integral $(31)_1$ is independent of $M_\circ$.

Consequently

$$\lim_{Q(\xi_1,\xi_2)\longrightarrow\infty}\oint_{\mathcal{L}_o} t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ, Q)\mathrm{d}s_{M_\circ} = \\ = \lim_{Q(\xi_1,\xi_2)\longrightarrow\infty} U_{\lambda\kappa}(Q)\underbrace{\oint_{\mathcal{L}_o} t_\lambda(M_\circ)\,\mathrm{d}s_{M_\circ}}_{\text{resultant}} = 0\,. \tag{34}$$

By repeating the line of thought leading to the asymptotic relations (33a,b,c), for the fundamental solutions of order two we obtain

$$T_{11}(M_\circ, Q) \approx -\frac{D}{\hat{R}}\left(\frac{\sqrt{\lambda_2}A_1}{\lambda_2\hat{n}_1^2+\hat{n}_2^2} - \frac{\sqrt{\lambda_1}A_2}{\lambda_1\hat{n}_1^2+\hat{n}_2^2}\right) = \frac{\tilde{T}_{11}(Q)}{\hat{R}}\,, \tag{35a}$$

$$T_{12}(M_\circ, Q) \approx -\frac{D}{\hat{R}}\left[\frac{1-\lambda_2}{\sqrt{\lambda_2}}\frac{A_2}{\lambda_2\hat{n}_1^2+\hat{n}_2^2} - \frac{1-\lambda_1}{\sqrt{\lambda_1}}\frac{A_1}{\lambda_1\hat{n}_1^2+\hat{n}_2^2}\right]\hat{n}_1\hat{n}_2 = \frac{\tilde{T}_{12}(Q)}{\hat{R}}\,, \tag{35b}$$

$$T_{21}(M_\circ, Q) \approx -\frac{D}{\hat{R}}\left[\frac{\sqrt{\lambda_1}\,(\lambda_1-1)\,A_2}{\lambda_1\hat{n}_1^2+\hat{n}_2^2} - \frac{\sqrt{\lambda_2}\,(\lambda_2-1)\,A_1}{\lambda_2\hat{n}_1^2+\hat{n}_2^2}\right]\hat{n}_1\hat{n}_2 = \frac{\tilde{T}_{21}(Q)}{\hat{R}}\,, \tag{35c}$$

$$T_{22}(M_\circ, Q) \approx -\frac{D}{\hat{R}}\left(\frac{\sqrt{\lambda_2}A_1}{\lambda_1\hat{n}_1^2+\hat{n}_2^2} - \frac{\sqrt{\lambda_1}A_2}{\lambda_2\hat{n}_1^2+\hat{n}_2^2}\right) = \frac{\tilde{T}_{22}(Q)}{\hat{R}}\,, \tag{35d}$$

where $\tilde{T}_{\lambda\kappa} = \tilde{T}_{\lambda\kappa}(Q)$ is defined by the above equations. Since for large $\hat{R}$ the kernel $T_{\lambda\kappa}(M_\circ, Q)$ is independent of $M_\circ$ and tends to zero, it follows that

$$\lim_{Q(\xi_1,\xi_2)\longrightarrow\infty}\oint_{\mathcal{L}_o} u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ, Q)\mathrm{d}s_{M_\circ} = \\ = \lim_{Q(\xi_1,\xi_2)\longrightarrow\infty}\frac{\tilde{T}_{\lambda\kappa}(Q)}{\hat{R}}\oint_{\mathcal{L}_o} u_\lambda(M_\circ)\,\mathrm{d}s_{M_\circ} = 0\,. \tag{36}$$

This verifies that the asymptotic behavior of the displacement representation (26) is as expected.

6 Examples

In this section we solve two external boundary value problems. First we consider the plane with a circular hole (Figure 3.b.); second the plane with a rigid inclusion (Figure 3.c.). The material is birch for which $s_{11} = 8.497 \times 10^{-5}$, $s_{12} = s_{21} = -6.11 \times 10^{-2}$, $s_{22} = 1.6999 \times 10^{-4}$ and $s_{66} = 1.456 \times 10^{-3}$. We assume that $\tau_{12}(\infty) = \sigma_{22}(\infty) = 0$ and $\sigma_{11}(\infty) = p = \text{constant}$. For completeness Figure 3.a. shows the region to use if we solve the integral equation of the direct method in its traditional form, i.e., if the exterior region is replaced by a bounded one.

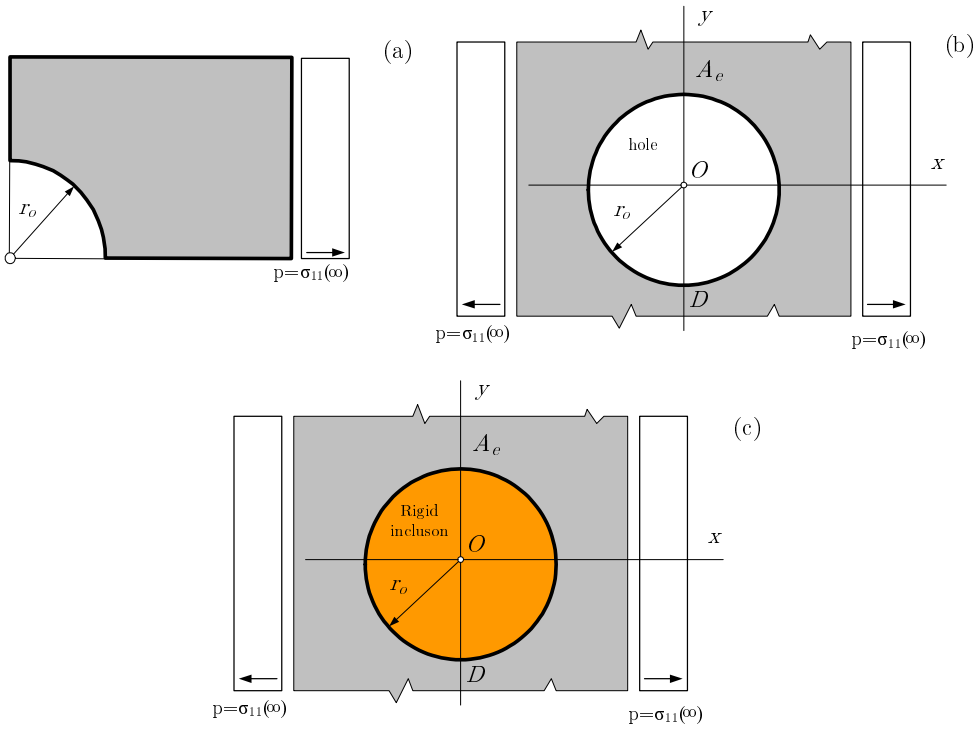

Figure 3:

Lekhtniski's book [9] contains closed form solutions for the stresses on the boundary, as well as numerical values which can be found in Table 17. on page 197. In this paper we show the results as computed by solving integral equation (27) (typeset in red) and the results taken from [9] – see Tables 1. and 2. We used a polar coordinate system, and the tables contain the quotients σ_θ/p for the plane with circular hole and σ_r/p $\tau_{r\theta}/p$, σ_θ/p for the plane with the circular inclusion.

	Circular hole	
Polar angle	σ_θ/p	[9], p.197
0°	−0.7071	−0.707
15°	−0.3399	−0.340
30°	0.0692	0.069
45°	0.4040	0.404
60°	0.9644	0.966
75°	2.5771	2.577
90°	5.4514	5.453

Table 1: Results for the circular hole

	Rigid kernel					
Polar angle	σ_r/p	[9], p.197	$\tau_{r\theta}/p$	[9], p.197	σ_θ/p	[9], p.197
0°	1.2363	1.237	0.0000	0.000	0.0445	0.044
15°	1.1562	1.156	−0.2994	−0.299	0.0934	0.093
30°	0.9370	0.937	−0.5185	−0.519	0.2697	0.270
45°	0.6378	0.698	−0.5987	−0.599	0.5154	0.516
60°	0.3383	0.388	−0.5185	−0.519	0.6989	0.699
75°	0.1192	0.119	−0.2994	−0.299	0.5637	0.564
90°	0.0389	0.039	0.0000	0.000	0.0028	0.003

Table 2: Results for the rigid inclusion

We used partially discontinuous elements of order two. The results obtained are in good agreement with those taken from Lekhtniski's book [9].

7 Concluding remarks

For the sake of completeness, note that [10] by Constanda gives an asymptotic expansion for the displacements at infinity which ensures the validity of the Betti formula for exterior regions. Under this condition the total strain energy stored in the region is bounded. In addition uniqueness and existence proofs are easy to give.

We have modified the Somigliana formulas for exterior regions by assuming that the strains are constants and accordingly the displacements are linear at infinity. Under this condition the strain energy density is bounded (but not necessarily the strain energy), and there is no need to replace the exterior region by a finite one if a constant stress condition is prescribed at infinity. This can be an advantage if one considers an infinite plane with holes or cracks in it and the plane is subjected to constant stresses at infinity and an attempt is made to determine the stresses in finite. It is easy to modify existing codes to perform computations. Results for two simple problem are also presented.

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A Formulae for strains

Making use of equations (13), (14) and (26) from the kinematic equations (1) we obtain

$$e_{\alpha\beta} = \oint_{\mathcal{L}_o} t_\lambda(M_\circ)\mathcal{D}_{\lambda\alpha\beta}(M_\circ,Q)\,\mathrm{d}s_{M_\circ} - \oint_{\mathcal{L}_o} u_\lambda(M_\circ)\mathcal{S}_{\lambda\alpha\beta}(M_\circ,Q)\,\mathrm{d}s_{M_\circ} \qquad Q\in A_e \tag{37}$$

where

$$\begin{aligned}
&\mathcal{D}_{\lambda 11} = U_{\lambda 1}\,\partial_1\ , && \mathcal{S}_{\lambda 11} = T_{\lambda 1}\,\partial_1\ ,\\
&\mathcal{D}_{\lambda 12} = \frac{1}{2}\left(U_{\lambda 2}\,\partial_1 + U_{\lambda 1}\,\partial_2\right) = \mathcal{D}_{\lambda 21}\ , && \mathcal{S}_{\lambda 12} = \frac{1}{2}\left(T_{\lambda 2}\,\partial_1 + T_{\lambda 1}\,\partial_2\right) = \mathcal{D}_{\lambda 21}\ ,\\
&\mathcal{D}_{\lambda 22} = U_{\lambda 2}\,\partial_2\ , && \mathcal{S}_{\lambda 22} = T_{\lambda 2}\,\partial_2\ .
\end{aligned} \tag{38}$$

If we introduce the notations

$$\begin{aligned}
&b_1 = \frac{\sqrt{\lambda_1}}{\rho_1^2}\ , \quad b_2 = \frac{\sqrt{\lambda_2}}{\rho_2^2}\ , \quad c = \sqrt{\lambda_1}\sqrt{\lambda_2}r_1^2 + r_2^2\ , \quad d = (\sqrt{\lambda_1} - \sqrt{\lambda_2})r_2\ ,\\
&k_1 = \frac{\lambda_1^{\frac{3}{2}}A_1}{\rho_1^2}\ , \quad k_2 = \frac{\lambda_2^{\frac{3}{2}}A_2}{\rho_2^2}\ , \quad f_1 = \frac{A_1}{(\lambda_1^{\frac{3}{2}}r_1^2 + r_2^2)^2}\ , \quad f_2 = \frac{A_2}{(\lambda_2^{\frac{3}{2}}r_1^2 + r_2^2)^2}
\end{aligned} \tag{39}$$

for the derivatives in equation (38) we have

$$\begin{aligned}
&\partial_1 U_{11} = D\left(b_2\lambda_2 A_1^2 r_1 - b_1\lambda_1 A_2^2 r_1\right)\ ,\\
&\partial_1 U_{12} = \frac{DA_1A_2}{1+d^2r_1^2}\left(2\sqrt{\lambda_1}\sqrt{\lambda_2}dr_1^2 - cd\right) = \partial_1 U_{21}\ ,\\
&\partial_1 U_{22} = D\left(b_1 A_1^2 r_1 - b_2 A_2^2 r_1\right)\ ,\\
&\partial_2 U_{11} = D\left(\ b_2 A_1^2 r_2 - b_1 A_2^2 r_2\right)\ ,\\
&\partial_2 U_{12} = \frac{DA_1A_2}{1+d^2r_1^2}\left[2dr_1r_2 - c(\sqrt{\lambda_1} - \sqrt{\lambda_2})r_1\right] = \partial_2 U_{21}\ ,\\
&\partial_2 U_{22} = -D\left(\frac{A_2^2 r_2}{\rho_2^2\sqrt{\lambda_2}} - \frac{A_1^2 r_2}{\rho_1^2\sqrt{\lambda_1}}\right)
\end{aligned} \tag{40}$$

and

$$\begin{aligned}
\partial_1 T_{11} &= D\left[\frac{2b_2\lambda_2 A_1 r_1}{\rho_2^2} - \frac{2b_1\lambda_1 A_2 r_1}{\rho_1^2}\right](r_1n_1 + r_2n_2) - D\left[b_2A_1 - b_1A_2\right]n_1\ ,\\
\partial_1 T_{12} &= D\left[\frac{2b_1\lambda_1 A_1 r_1}{\rho_1^2} - \frac{2b_2\lambda_2 A_2 r_1}{\rho_2^2}\right] r_1n_2 - D\left[b_1A_1 - b_2A_2\right]n_2 -\\
&\quad - D\left[2f_1\lambda_1 r_1 - 2f_2\lambda_2 r_1\right] r_2n_1\ ,\\
\partial_1 T_{21} &= D\left[\frac{2b_1\lambda_1^2 A_2 r_1}{\rho_1^2} - \frac{2b_1\lambda_2^2 A_1 r_1}{\rho_2^2}\right] r_1n_2 - D\left[b_1\lambda_1A_2 - b_2\lambda_2A_1\right]n_2 -\\
&\quad - D\left[\frac{2b_1\lambda_1 A_2 r_1}{\rho_1^2} - \frac{2b_2\lambda_2 A_1 r_1}{\rho_2^2}\right] r_2n_1\ ,\\
\partial_1 T_{22} &= D\left[\frac{2b_1\lambda_1 A_1 r_1}{\rho_1^2} - \frac{2b_2\lambda_2 A_2 r_1}{\rho_2^2}\right](r_1n_1 + r_2n_2) - D\left[b_1A_1 - b_2A_2\right]n_1\ ,
\end{aligned} \tag{41a}$$

$$
\begin{aligned}
\partial_2 T_{11} &= D\left[\frac{b_2 A_1 2r_2}{\rho_2^2} - \frac{b_1 A_2 2r_2}{\rho_1^2}\right](r_1 n_1 + r_2 n_2) - D\left[\, b_2 A_1 - \; b_1 A_2\right] n_2 \;, \\
\partial_2 T_{12} &= D\left[\frac{b_1 A_1 2r_2}{\rho_1^2} - \frac{b_2 A_2 2r_2}{\rho_2^2}\right] r_1 n_2 - D\left[2 f_1 r_2 - 2 f_2 r_2\right] r_2 n_1 + \\
&\quad + D\left[f_1 - f_2\right] n_1 \;, \\
\partial_2 T_{21} &= D\left[\frac{b_1 \lambda_1 A_2 2r_2}{\rho_1^2} - \frac{b_2 \lambda_2 A_1 2r_2}{\rho_2^2}\right] r_1 n_2 - D\left[\frac{b_1 A_2 2r_2}{\rho_1^2} - \frac{b_2 A_1 2r_2}{\rho_2^2}\right] r_2 n_1 + \\
&\quad + D\left[b_1 A_2 - b_2 A_1\right] n_1 \;, \\
\partial_2 T_{22} &= D\left[\frac{b_1 A_1 2r_2}{\rho_1^2} - \frac{b_2 A_2 2r_2}{\rho_2^2}\right](r_1 n_1 + r_2 n_2) - D\left[b_1 A_1 - b_2 A_2\right] n_2 \;.
\end{aligned}
\tag{41b}
$$